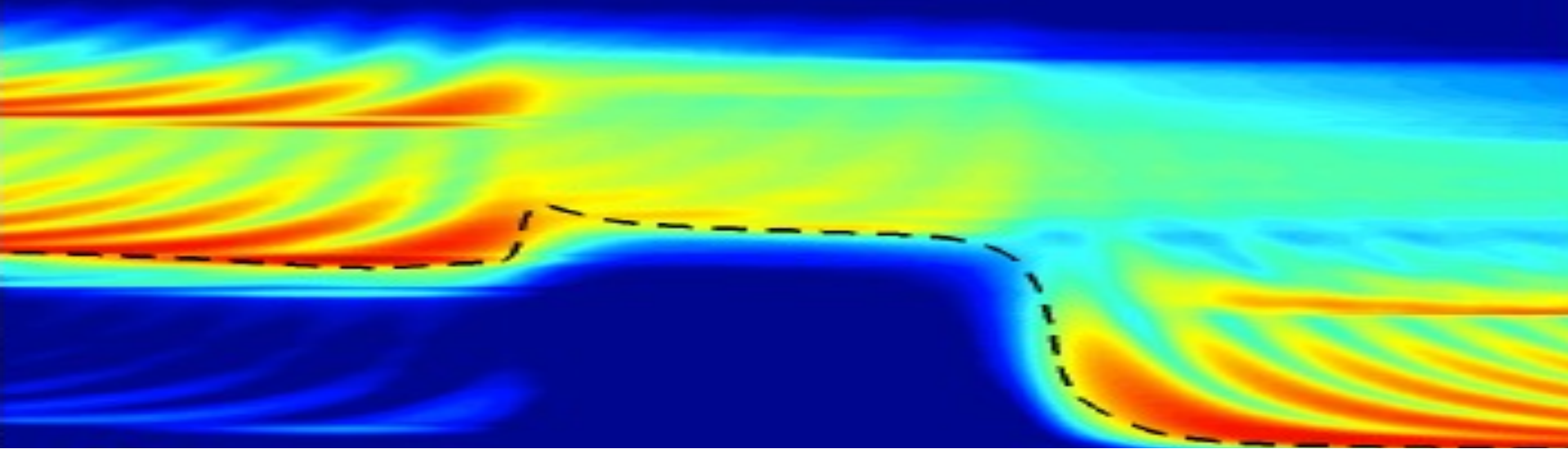

EECS 16B

Designing Information Devices and Systems II

Lecture 15

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Outline

- System Stability (Recap)
- Stabilization by Feedback
- Control Canonical Form

$x[i+1] = \lambda x[i] + e[i] \qquad |\lambda| < 1$

System Stability (Continuous Time)

Stability for the Scalar Case: $\frac{d}{dt}x(t) = \lambda_c x(t) + w(t)$

$$\frac{x[i+1]-x[i]}{\Delta} = \lambda_c x[i] + w[i]$$

$x[i+1] = (1+\lambda_c\Delta)x[i] + \Delta w[i]$

$(1+\lambda_c\Delta) < 1 \qquad \lambda_d < 1$

$\lambda_c < 0$

$$x(t) = e^{\lambda t}x(0) + \int_0^t e^{\lambda(t-\tau)}w(\tau)d\tau$$

$$\left|\int_0^t e^{\lambda(t-\tau)}w(\tau)d\tau\right| \leq \int_0^t e^{\lambda(t-\tau)}d\tau M = \frac{e^{\lambda t}-1}{\lambda}M$$

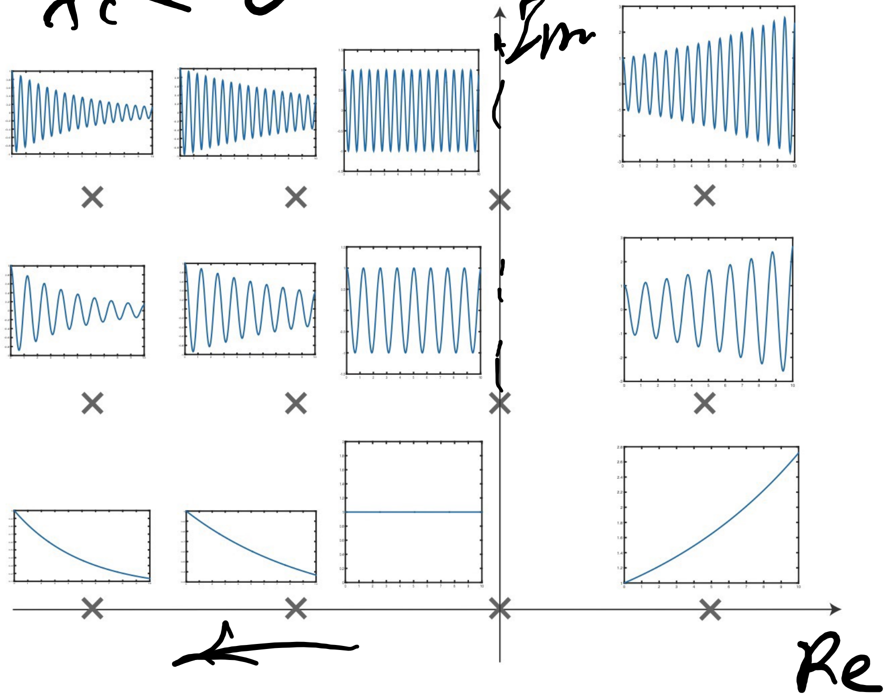
$x(t)$ bounded $\leftarrow$ $e^{\lambda t}$ bounded?

$e^{\lambda t}$

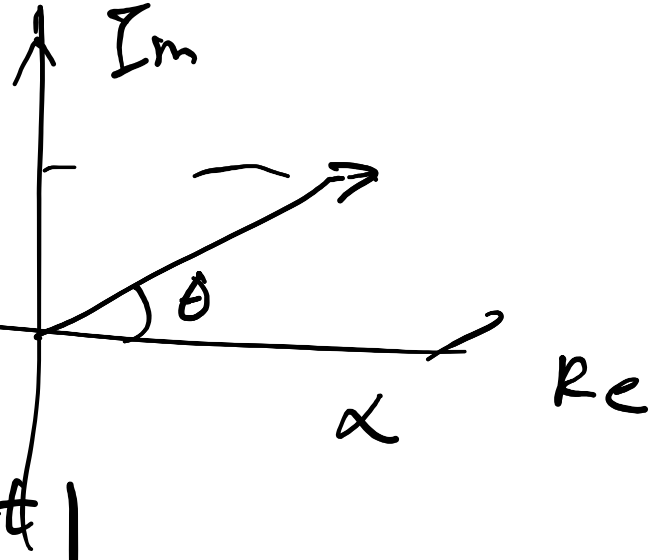
$\lambda > 0 \quad e^{\lambda t} \nearrow \infty$ as $t \nearrow \infty$

$\lambda = 0 \quad t \cdot M \nearrow \infty$ as $t \to \infty$

$\lambda < 0 \quad e^{\lambda t} \searrow 0$



$\lambda = \underline{\alpha} + j\underline{\theta}$



$$e^{\lambda t} = e^{\alpha t} \cdot e^{j\theta t}$$

$$|e^{\lambda t}| = \underline{|e^{\alpha t}|} \cdot \underbrace{|e^{j\theta t}|}_{=1}$$

$$e^{\underline{Re(\lambda)}\, t}$$

$Re(\lambda) < 0$

System Stability (Continuous Time)

Stability for the Vector Case: $\dot{\vec{x}}(t) = A\vec{x}(t) + \vec{w}(t) \quad \in \mathbb{R}^n$

Diagonalize or triangularize: $T = V^{-1}AV \quad \vec{z} = V^{-1}\vec{x}$ $\quad \vec{x} = V\vec{z}$

$$\dot{\vec{z}}(t) = V^{-1}AV\vec{z}(t) + V^{-1}\vec{w}(t)$$

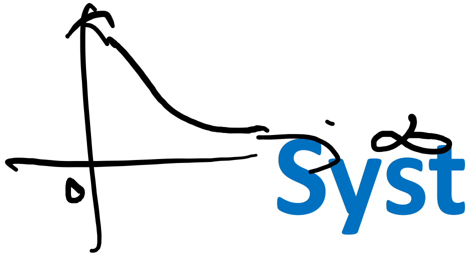
$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \vdots \\ \dot{z}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ & \lambda_2 & & \vdots \\ & & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} + V^{-1}\vec{w}(t)$$

$\lambda_n < 0$

$$\dot{z}_{n-1}(t) = \lambda_{n-1} z_{n-1}(t) + * z_n(t) + (V^{-1}w)_{n-1}$$

$$\mathrm{Re}(\lambda_k) < 0 \qquad k = 1, 2, \ldots, n.$$

$e^{\lambda t}$



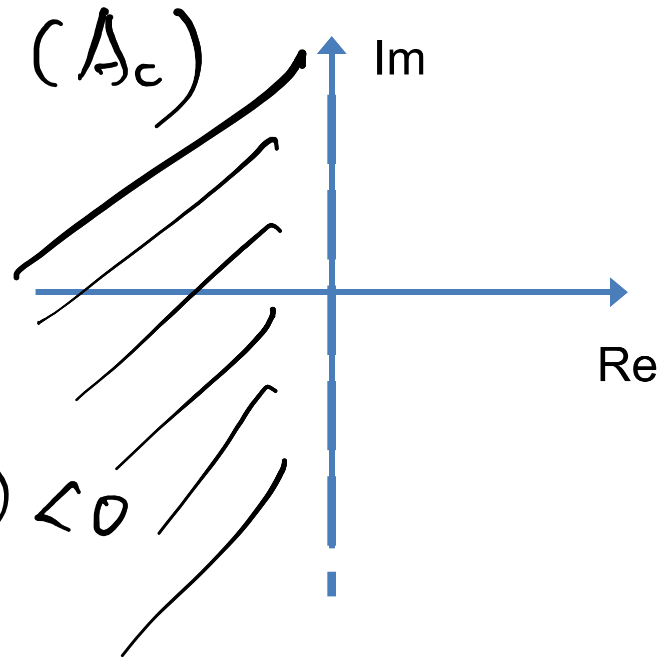
System Stability (Recap)

Definition: We say a system is *bounded input bounded state (BIBS) stable* if its state stays bounded, $\forall i \ \|\vec{x}[i]\| \leq C$, for any initial condition, any bounded input, and bounded disturbance.

Continuous time: $\dot{\vec{x}}(t) = A_c\vec{x}(t) + \vec{w}(t) \ \in \mathbb{R}^n$

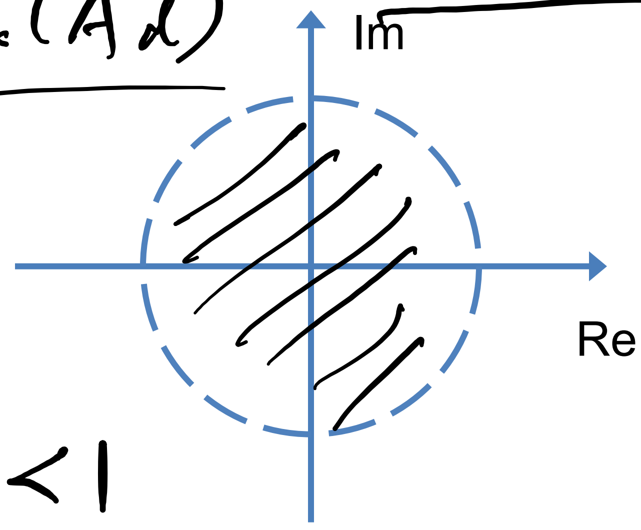
Discrete time: $\vec{x}[i+1] = A_d\vec{x}[i] + \vec{e}[i] \in \mathbb{R}^n$

$\lambda_k(A_c)$



$Re(\lambda_k) < 0$

$\lambda_k(A_d)$

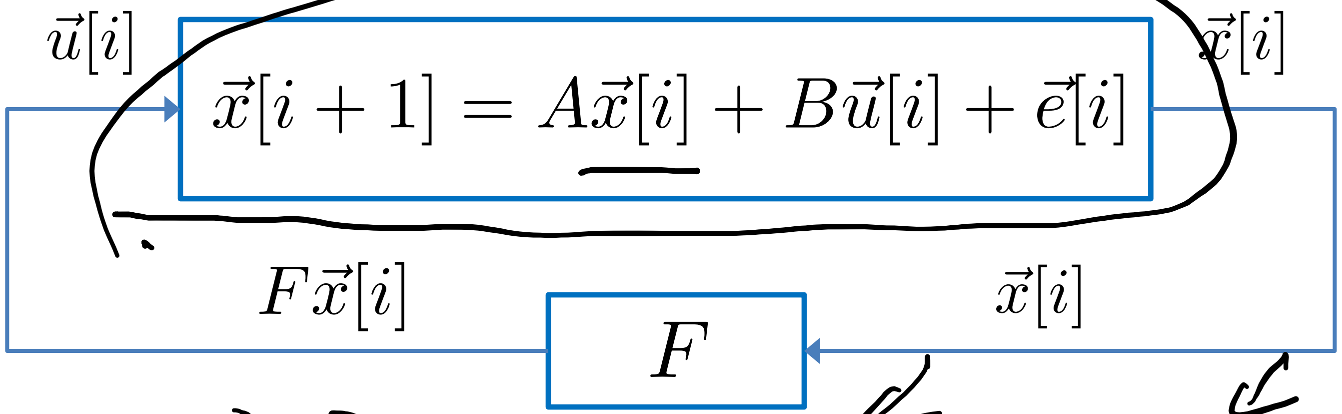


$|\lambda_k| < 1$

System Stabilization

$$\vec{x}[i+1] = \underline{A\vec{x}[i]} + \underline{B\vec{u}[i]} + \vec{e}[i] \in \mathbb{R}^n$$

What if some or all eigenvalues of A are outside of the unit circle? Consider the feedback: $\vec{u}[i] = F\vec{x}[i]$



$\rightarrow \vec{u}[i] = F\vec{x}[i]$

$u[i] = f_1 x_1[i] + f_2 x_2[i] + \cdots + f_n x_n[i]$

$\vec{x}[i+1] = A\vec{x}[i] + BF\vec{x}[i] + \vec{e}[i]$

$\rightarrow \vec{x}[i+1] = [A+BF]\vec{x}[i] + \vec{e}[i]$

$$A_{cl} = A + BF$$

System Stabilization (Example 1)

Scalar case: $x[i+1] = 3x[i] + u[i] + e[i]$

$3^{\ell} \nearrow$

$u[i] = f\,x[i]$

$x[i+1] = \underbrace{(3+f)}_{\lambda_{cl}}\,x[i] + e[i]$

$|\lambda_{cl}| < 1 \Rightarrow |(3+f)| < 1$

$-1 < 3+f < 1$

$f \in (-4, -2)$

System Stabilization (Example 2)

Vector case: $\vec{x}[i+1] = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{e}[i]$

$$u[i] = f_1 x_1[i] + f_2 x_2[i]$$

$$\vec{x}[i+1] = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [f_1, f_2] \vec{x}[i] + \vec{e} \qquad = [f_1, f_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 3+f_1 & 1+f_2 \\ 0 & -2 \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix} + \vec{e} \qquad \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix}$$

$$\lambda_1 = 3 + f_1$$

$$\lambda_2 = -2$$

System Stabilization (Example 3)

Vector case: $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i] + \vec{e}[i]$

$[f_1 \;\; f_2]\vec{x}$

A

$$\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3+f_1 & -2+f_2 \end{bmatrix} \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix} + \vec{e}$$

$\det(\lambda I - A_c)$?

$\det(\lambda I - A)$

$= \det\begin{pmatrix} \lambda & -1 \\ -3 & \lambda+2 \end{pmatrix} = \lambda^2 + 2\lambda - 3$

$= (\lambda+3)(\lambda-1)$

$\lambda_1 = 1$

$\lambda_2 = -3$

$u = f_1 x_1 + f_2 x_2$

System Stabilization (Example 3)

Vector case: $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i] + \vec{e}[i]$

$$\det(\lambda I - A_{cl}) = \det \begin{bmatrix} \lambda & -1 \\ -3-f_1 & \lambda+2-f_2 \end{bmatrix}$$

$$= \lambda^2 + \lambda(2-f_2) - (3+f_1)$$

$$= (\lambda-\lambda_1)(\lambda-\lambda_2) \quad ?$$

$$= \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$$

$\lambda_1 = -0.5$, $\lambda_2 = -0.3$

$$\begin{cases} -(\lambda_1+\lambda_2) = 2-f_2 \\ \lambda_1 \cdot \lambda_2 = -(3+f_1) \end{cases}$$

$$\begin{cases} f_1 = -\lambda_1\lambda_2 - 3 \\ f_2 = 2 + (\lambda_1+\lambda_2) \end{cases}$$

Controllable Canonical Form

Single input case: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \in \mathbb{R}^n \qquad A_{cl} = A + BF$

$$A = \begin{array}{cc} \begin{matrix} x_1 \\ \\ \\ x_{n-1} \\ x_n \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots\ 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix} \end{array}, \quad u \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$F = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \end{bmatrix}$$

Controllable Canonical Form

Characteristic Polynomial of A is simple:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda & -1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & \lambda - a_n \end{bmatrix}$$

$$\det(\lambda I - A) = \boxed{\lambda^n - a_n \lambda^{n-1} - a_{n-1}\lambda a^{n-2} - \cdots - a_2\lambda - a_1}$$

$$A \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix}$$

$$\det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -a_1 & -a_2 & \lambda - a_3 \end{bmatrix}$$

check 3x3

$\cancel{a_n} \quad -a_{n-1}$

Controllable Canonical Form

$$\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \;\; \in \mathbb{R}^n \qquad A_{cl} = A + BF$$

$$u = f_1 x_1 + f_2 x_2 + \cdots + f_n x_n$$

$$A_{cl} = A + BF = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \underbrace{\begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \end{bmatrix}}_{F}$$

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & 0 & & 0 & 1 \\ a_1+f_1 & a_2+f_2 & \cdots & & a_n+f_n \end{bmatrix}$$

$$\det(\lambda I - A_{cl})$$

$$= \lambda^n - \underline{(a_n+f_n)}\lambda^{n-1} - \underline{(a_{n-1}+f_{n-1})}\lambda^{n-2} - \cdots - \underline{(a_1+f_1)}$$

$$= (\lambda - \underline{\lambda_1})(\lambda - \underline{\lambda_2}) \cdots (\lambda - \underline{\lambda_n})$$

$$(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$

$$= \lambda^n - (\lambda_1+\lambda_2\cdots+\lambda_n)\lambda^{n-1} + \cdots + (-1)^n\lambda_1\lambda_2\cdots\lambda_n$$

$$\lambda^n - (a_n+f_n)\lambda^{n-1} - \cdots - (a_1+f_1)$$

$$f_n = \lambda_1+\cdots+\lambda_n - a_n \quad \cdots \quad f_1 = (-1)^n\lambda_1\lambda_2\cdots\lambda_n - a_1$$

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$

Can we bring the system to the canonical form via a similarity transform: $\vec{z} = T\vec{x}$?

$(A, B) \rightarrow \{A_{canonical}, B_{canonical}\}$

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \;\; \in \mathbb{R}^n$

Claim: we can convert the above system to the canonical form if the following **controllability matrix:**

$$\mathcal{C} \doteq [A^{n-1}B \mid \cdots | AB \mid B] \;\; \in \mathbb{R}^{n\times n} \text{ is invertible.}$$

$$\vec{z}[i+1] = TAT^{-1}\vec{z}[i] + TBu[i] + T\vec{e}[i] \;\; \in \mathbb{R}^n \qquad \vec{z} = T\vec{x}$$

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a'_1 & a'_2 & \cdots & a'_{n-1} & a'_n \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$

Claim: we can convert the above system to the canonical form if the following **controllability matrix:**

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$$\vec{z}[i+1] = TAT^{-1}\vec{z}[i] + TBu[i] + T\vec{e}[i] \ \in \mathbb{R}^n \qquad \vec{z} = T\vec{x}$$

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a'_1 & a'_2 & \cdots & a'_{n-1} & a'_n \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$

$$\vec{z}[i+1] = TAT^{-1}\vec{z}[i] + TBu[i] + T\vec{e}[i] \ \in \mathbb{R}^n \qquad \vec{z} = T\vec{x}$$

$$\vec{z}[i+1] = A_z\vec{x}[i] + B_z u[i] + \vec{e}'[i]$$

$$u[i] = F_z\vec{z}[i] = F_z T\vec{x}[i]$$

Claim: the closed loop system $A + BF = A + BF_zT$ has the same eigenvalues as $A_z + B_zF_z$

Feedback Control (Summary)

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$

- It is possible to stabilize the system via state **feedback control**:

$$\vec{u}[i] = F\vec{x}[i]$$

- When is this possible? The system is **controllable**:

$$\mathcal{C} \doteq [A^{n-1}B \mid \cdots \mid AB \mid B] \ \in \mathbb{R}^{n\times n} \text{ is invertible.}$$

- How to design eigenvalues of closed-loop system (to stabilize)? Controllable **canonical form**:

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a'_1 & a'_2 & \cdots & a'_{n-1} & a'_n \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$