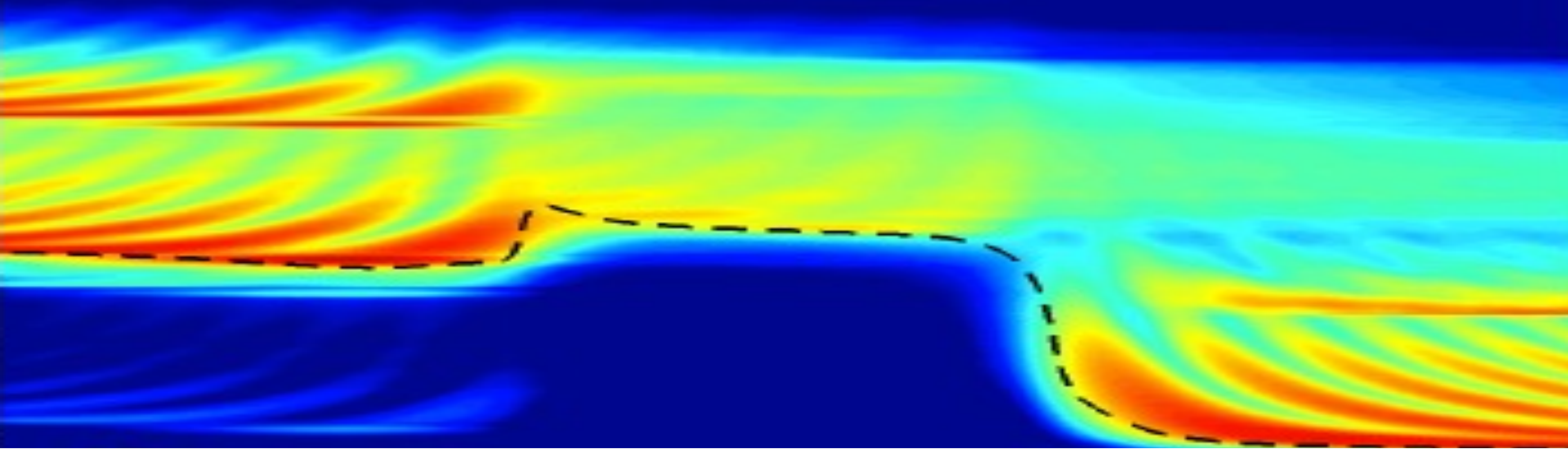

EECS 16B
Designing Information Devices and Systems II
Lecture 16

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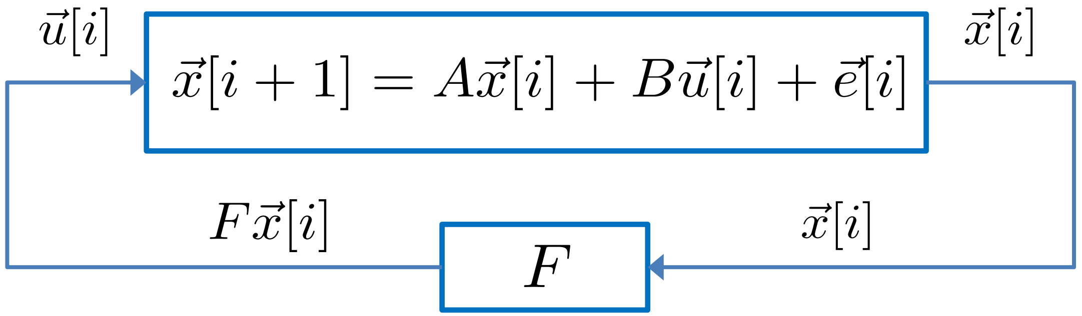
Outline

- Controllable Canonical Form
- Feedback Stabilization
- Controllability

Feedback Control (Summary)

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \;\; \in \mathbb{R}^n$

- It is possible to stabilize the system via state feedback control:



- We know how to do this if the system is in the following canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad F = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \end{bmatrix}$$

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$

$$\vec{z}[i+1] = TAT^{-1}\vec{z}[i] + TBu[i] + T\vec{e}[i] \ \in \mathbb{R}^n \qquad \vec{z} = T\vec{x}$$

$$\vec{z}[i+1] = A_z\vec{x}[i] + B_z u[i] + \vec{e}'[i]$$

$$u[i] = F_z\vec{z}[i] = F_z T\vec{x}[i]$$

Claim: the closed loop system $A + BF = A + BF_zT$ has the same eigenvalues as $A_z + B_zF_z$

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$

If there exists a transformation, $\vec{z} = T\vec{x}$ such that:

$$\vec{z}[i+1] = TAT^{-1}\vec{z}[i] + TBu[i] + T\vec{e}[i] \ \in \mathbb{R}^n$$

has the canonical form:

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a'_1 & a'_2 & \cdots & a'_{n-1} & a'_n \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Claim: we can convert the above system to the canonical form if the following **controllability matrix:**

$$\mathcal{C} \doteq [A^{n-1}B \mid \cdots | AB \mid B] \ \in \mathbb{R}^{n \times n} \text{ is invertible.}$$

Controllable Canonical Form

Claim: we can convert the above system to the canonical form if the following **controllability matrix:**

$$\mathcal{C} \doteq [A^{n-1}B \mid \cdots \mid AB \mid B] \in \mathbb{R}^{n \times n} \text{ is invertible.}$$

Proof:

Controllable Canonical Form

Claim: we can convert the above system to the canonical form if the following **controllability matrix:**

$$\mathcal{C} \doteq [A^{n-1}B \mid \cdots \mid AB \mid B] \in \mathbb{R}^{n\times n} \text{ is invertible.}$$

Proof continued:

Controllable Canonical Form (Example)

Stabilize the following system with feedback: $\vec{x}[i+1] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$

Controllable Canonical Form (Example)

Convert this system to the canonical form: $\vec{x}[i+1] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$

Controllable Canonical Form (Example)

Can you stabilize the following system with feedback or convert it to the canonical form?

$$\vec{x}[i+1] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$$

Controllability

Given a system $\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ starting from $\vec{x}[0]$, can we bring the state to any target final state $\vec{x}_f \in \mathbb{R}^n$ at some time $i = \ell$?

Controllability

Definition: a system $\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ is said to be ***controllable*** if given any target state $\vec{x}_f \in \mathbb{R}^n$ and initial state $\vec{x}[0]$, we can find a time $i = \ell$ and a sequence of control input $u[0], \ldots, u[\ell]$ such that $\vec{x}[\ell] = \vec{x}_f$

Controllability (Examples)

Definition: a system $\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ is said to be ***controllable*** if given any target state $\vec{x}_f \in \mathbb{R}^n$ and initial state $\vec{x}[0]$, we can find a time $i = \ell$ and a sequence of control input $u[0], \ldots, u[\ell]$ such that $\vec{x}[\ell] = \vec{x}_f$

Controllability

Lemma: Consider $\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots | AB \mid B] \;\in \mathbb{R}^{n\times\ell}$

If $\text{rank}[\mathcal{C}_{\ell+1}] = \text{rank}[\mathcal{C}_\ell]$ then $\text{rank}[\mathcal{C}_m] = \text{rank}[\mathcal{C}_\ell]$ for all $m \geq \ell + 1$

Proof:

Controllability

Lemma: Consider $\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots |AB \mid B] \;\in \mathbb{R}^{n\times\ell}$

If $\mathrm{rank}[\mathcal{C}_{\ell+1}] = \mathrm{rank}[\mathcal{C}_\ell]$ then $\mathrm{rank}[\mathcal{C}_m] = \mathrm{rank}[\mathcal{C}_\ell]$ for all $m \geq \ell + 1$