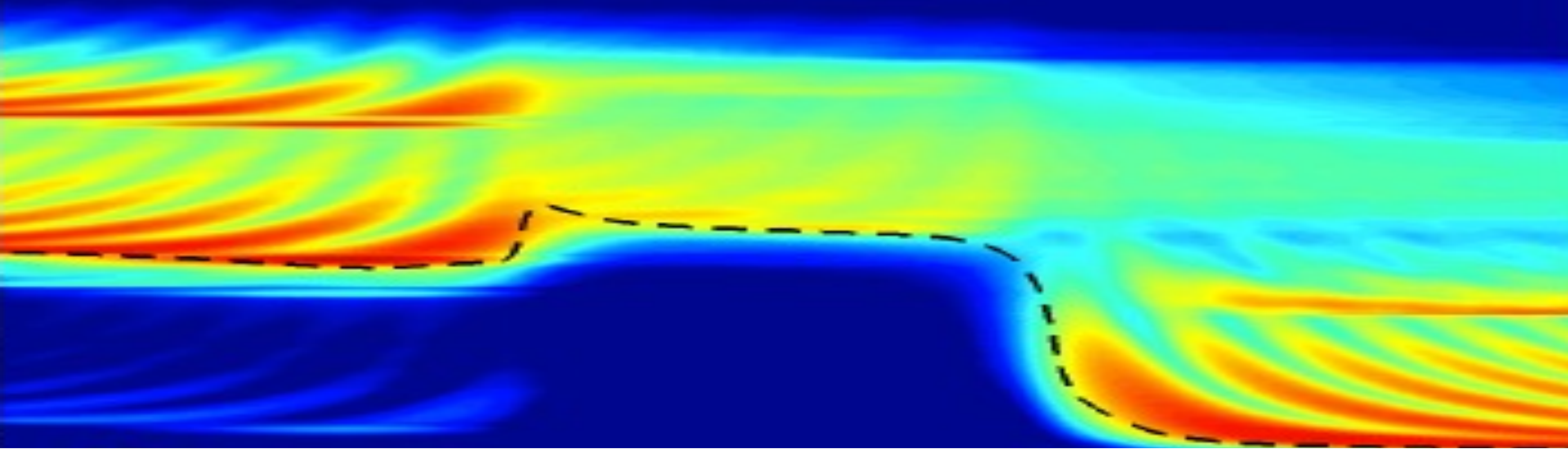


EECS 16B
Designing Information Devices and Systems II
Lecture 16

Prof. Yi Ma

Department of Electrical Engineering and Computer Sciences, UC Berkeley,
yima@eecs.berkeley.edu

Outline

- Controllable Canonical Form
- Feedback Stabilization
- Controllability

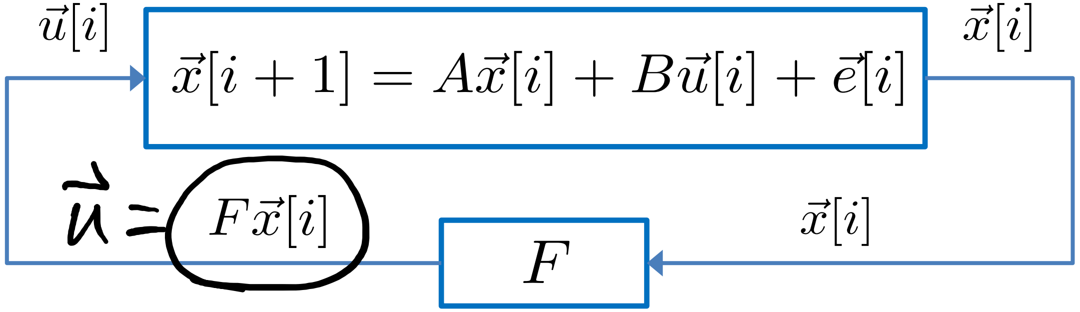
Feedback Control (Summary)

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \quad \in \mathbb{R}^n$

$$\vec{x}[i+1] = \underbrace{[A+BF]}_{A_{cl}}\vec{x}[i] + \vec{e}[i]$$

- It is possible to stabilize the system via state feedback control:

$\vec{y} = C\vec{x}$



$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & 0 & 1 \\ a_1+f_1 & a_2+f_2 & \cdots & & a_n+f_n \end{bmatrix}$$

- We know how to do this if the system is in the following canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad F = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \end{bmatrix}$$

$A + BF$

Controllable Canonical Form

$M\vec{v} = \lambda \vec{v}$

$TMT^{-1}(T\vec{v})$

$= TM\vec{v}$

$= T\lambda\vec{v}$

$= \lambda(T\vec{v})$

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \in \mathbb{R}^n$

$\vec{z} = T\vec{x}$ $\quad \vec{z}[i+1] = \underbrace{TAT^{-1}}_{A_z}\vec{z}[i] + \underbrace{TB}_{B_z}u[i] + T\vec{e}[i] \in \mathbb{R}^n$

$\vec{z}[i+1] = A_z\vec{z}[i] + B_z u[i] + \vec{e}'[i]$ $\quad u[i] = F_z\vec{z}[i] = \underbrace{F_z T}_{F}\vec{x}[i]$

$A_{z\,cl} = A_z + B_z F_z$

$A_{cl} = A + BF$?

Claim: the closed loop system $A + BF = A + BF_zT$ has the same eigenvalues as $A_z + B_zF_z$

$T(A+BF)T^{-1}$

$= TAT^{-1} + TBFT^{-1}$

$= A_z + B_zF_z$

M has the same eigenvalues of TMT^{-1}

$\det(\lambda I - M) = \det(T)\det(\lambda I - M) \cdot \det(T^{-1})$

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$

If there exists a transformation, $\vec{z} = T\vec{x}$ such that:

$$\vec{z}[i+1] = \underbrace{TAT^{-1}}_{} \vec{z}[i] + TBu[i] + T\vec{e}[i] \ \in \mathbb{R}^n$$

A_z B_z

has the canonical form:

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a'_1 & a'_2 & \cdots & a'_{n-1} & a'_n \end{bmatrix}, \quad \underline{TB} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Claim: we can convert the above system to the canonical form if the following **controllability matrix:**

$T \rightarrow$ $$\mathcal{C} \doteq \underbrace{[A^{n-1}B \mid \cdots \mid AB \mid B]}_{} \in \mathbb{R}^{n\times n} \text{ is invertible.}$$

Controllable Canonical Form

Claim: we can convert the above system to the canonical form if the following **controllability matrix:**

$$\mathcal{C} \doteq [A^{n-1}B \mid \cdots \mid AB \mid B] \in \mathbb{R}^{n\times n} \text{ is invertible.}$$

$C^{-1} C = I$

Proof:

$$Q = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \underbrace{[A^{n-1}B, A^{n-2}B, \ldots, AB, B]}_{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \end{bmatrix} \quad Q$$

$$\begin{bmatrix} q^T B \\ q^T AB \\ \vdots \\ q^T A^{n-1} B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad T = \begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}_{n\times n} A$$

$$TA = \begin{bmatrix} q^T A \\ q^T A^2 \\ \vdots \\ q^T A^n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ * & * & \cdots & \cdots & * \end{bmatrix}}_{(TAT^{-1})} \underbrace{\begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}}_{I}$$

Controllable Canonical Form

Claim: we can convert the above system to the canonical form if the following **controllability matrix:**

$$\mathcal{C} \doteq [A^{n-1}B \mid \cdots \mid AB \mid B] \in \mathbb{R}^{n\times n} \text{ is invertible.}$$

Proof continued:

$TA^{-1}T \leftarrow$ canonical form T invertible?

$$\underbrace{\begin{bmatrix} q^T \\ q^TA \\ \vdots \\ q^TA^{n-1} \end{bmatrix}}_{T} \underbrace{[A^{n-1}B, \dots AB, B]}_{\mathcal{C}} = \begin{bmatrix} q^TA^{n-1}B & \cdots & q^TAB & q^TB \\ q^TA^nB & q^TA^{n-1}B & \cdots & q^TAB \\ & \ddots & & \vdots \\ q^TA^{2n-2}B & \cdots & & q^TA^{n-1}B \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix} M$$

$T = \mathcal{C}^{-1}M \quad T^{-1} = M^{-1}\mathcal{C}$

Controllable Canonical Form (Example)

Convert this system to the canonical form: $\vec{x}[i+1] = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}[i] + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u[i]$

$$C = [A^{n-1}B, \ldots B]$$

$$\text{rank} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = 2$$

$$\begin{array}{cc} AB & B \end{array}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{C}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} q^T \\ * \end{bmatrix}$$

$$T = \begin{bmatrix} q^T \\ q^T A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$TAT^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

$$TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Controllable Canonical Form (Example)

Stabilize the following system with feedback: $\vec{x}[i+1] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$

$u = [f_1 x_1 + f_2 x_2]$

$\vec{z} = T\vec{x}$

$\vec{z}[i+1] = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \vec{z}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$

$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2$

$A + BF = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1, f_2]$

$= \begin{bmatrix} 1 & 1 \\ f_1 & 2+f_2 \end{bmatrix} = A_{cl}$

$\det(\lambda I - A_{cl})$

$= \det\begin{pmatrix} \lambda - 1 & -1 \\ -f_1 & \lambda - 2 - f_2 \end{pmatrix}$

$= \lambda^2 - (3 + f_2)\lambda + 2 + f_2 - f_1$

Controllable Canonical Form (Example)

Can you stabilize the following system with feedback or convert it to the canonical form?

$$\vec{x}[i+1] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$$

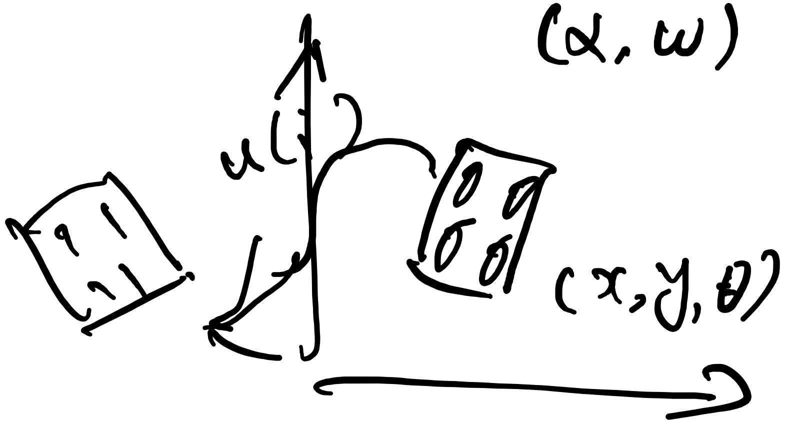
$n = 2$

$$[AB, B] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+f_1 & 1+f_2 \\ 0 & 2 \end{bmatrix}$$

Controllability

Given a system $\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ starting from $\vec{x}[0]$, can we bring the state to any target final state $\vec{x}_f \in \mathbb{R}^n$ at some time $i = \ell$?



$$\vec{x}[1] = A\vec{x}[0] + Bu[0]$$

$$\vec{x}[2] = A^2\vec{x}[0] + ABu[0] + Bu[1]$$

$$\vdots$$

$$\vec{x}[\ell] = A^\ell \vec{x}[0] + A^{\ell-1}Bu[0] + \cdots + Bu[\ell-1]$$

$$= A^\ell \vec{x}[0] + \underbrace{\underbrace{[A^{\ell-1}B, A^{\ell-2}B, \ldots, AB, B]}_{C_\ell} \underbrace{\begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[\ell-1] \end{bmatrix}}_{\vec{u}[\ell]}}_{\text{column space of } C_\ell}$$

span$[C_\ell]$

col$[C_\ell]$

$$\boxed{\ell \geq n}$$

Controllability

$\in \mathbb{R}^n$

Definition: a system $\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ is said to be ***controllable*** if given any target state $\vec{x}_f \in \mathbb{R}^n$ and initial state $\vec{x}[0]$, we can find a time $i = \ell$ and a sequence of control input $u[0], \ldots, u[\ell]$ such that $\vec{x}[\ell] = \vec{x}_f$

$$\vec{x}[\ell] = \vec{x}_f$$

$$\vec{x}[\ell] = A^\ell \vec{x}[0] + C_\ell \vec{u}[\ell] \qquad \vec{u}[\ell] = \begin{bmatrix} u[0] \\ \vdots \\ u[\ell-1] \end{bmatrix}$$

$$\underbrace{\vec{x}_f - A^\ell \vec{x}[0]}_{\Delta} \in \mathbb{R}^n$$

if $\boxed{\operatorname{span}(C_\ell) = \mathbb{R}^n}$

$$\Delta = C_\ell \cdot \vec{u}[\ell] \implies \vec{x}[\ell] = A^\ell \vec{x}[0] + \Delta$$

$$= \vec{x}_f$$

$$\operatorname{span}(C_\ell) \subseteq \mathbb{R}^n$$

$$A^\ell \vec{x}[0] + \operatorname{span}(C_\ell)$$

reachable space

Controllability (Examples)

Definition: a system $\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ is said to be ***controllable*** if given any target state $\vec{x}_f \in \mathbb{R}^n$ and initial state $\vec{x}[0]$, we can find a time $i = \ell$ and a sequence of control input $u[0], \ldots, u[\ell]$ such that $\vec{x}[\ell] = \vec{x}_f$

Controllability

Lemma: Consider $\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots | AB \mid B] \ \in \mathbb{R}^{n\times\ell}$

If $\operatorname{rank}[\mathcal{C}_{\ell+1}] = \operatorname{rank}[\mathcal{C}_\ell]$ then $\operatorname{rank}[\mathcal{C}_m] = \operatorname{rank}[\mathcal{C}_\ell]$ for all $m \geq \ell + 1$

Proof:

Controllability

Lemma: Consider $\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots \mid AB \mid B] \ \in \mathbb{R}^{n \times \ell}$

If $\operatorname{rank}[\mathcal{C}_{\ell+1}] = \operatorname{rank}[\mathcal{C}_\ell]$ then $\operatorname{rank}[\mathcal{C}_m] = \operatorname{rank}[\mathcal{C}_\ell]$ for all $m \geq \ell + 1$