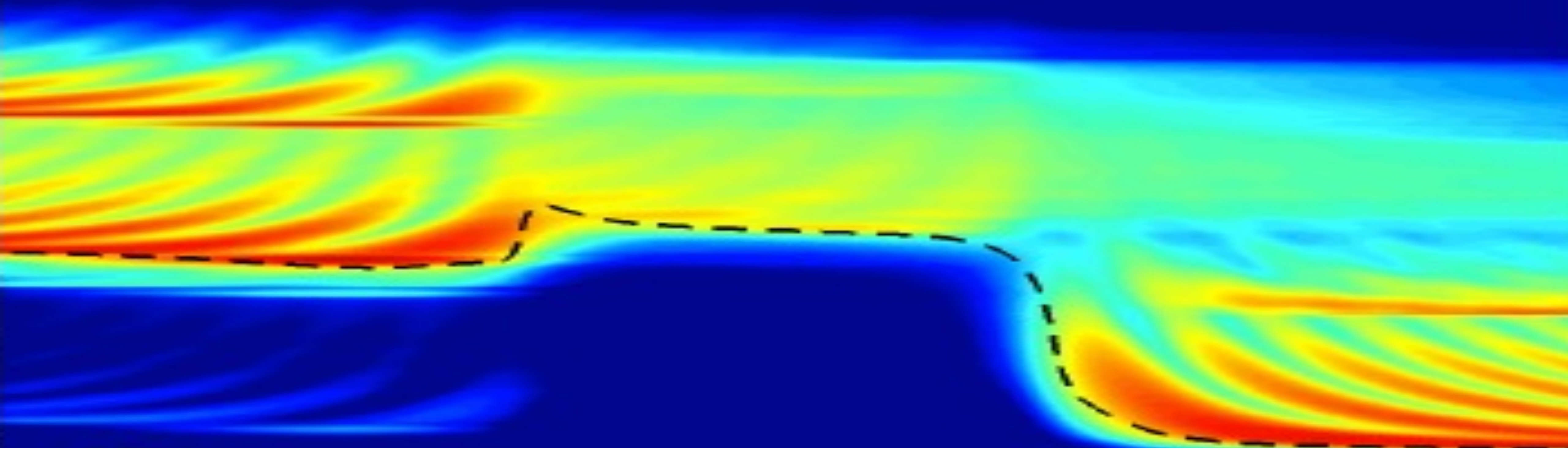

EECS 16B
Designing Information Devices and Systems II
Lecture 17

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Outline

- Condition for Controllability
- Orthonormal Bases
- Orthonormalization (Gram-Schmidt procedure)

Controllability

Definition: a system $\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ is said to be ***controllable*** if given any target state $\vec{x}_f \in \mathbb{R}^n$ and initial state $\vec{x}[0]$, we can find a time $i = \ell$ and a sequence of control input $u[0], \ldots, u[\ell]$ such that $\vec{x}[\ell] = \vec{x}_f$

$$\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots \mid AB \mid B] \in \mathbb{R}^{n \times \ell} \qquad \vec{x}[\ell] = A^\ell \vec{x}[0] + \mathcal{C}_\ell \vec{u}[\ell]$$

Condition for Controllability: $\text{span}[\mathcal{C}_\ell] = \mathbb{R}^n \quad \text{or} \quad \text{rank}[\mathcal{C}_\ell] = n$

Controllability

Lemma: Consider $\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots | AB \mid B] \ \in \mathbb{R}^{n\times\ell}$

If $\text{rank}[\mathcal{C}_{\ell+1}] = \text{rank}[\mathcal{C}_\ell]$ then $\text{rank}[\mathcal{C}_m] = \text{rank}[\mathcal{C}_\ell]$ for all $m \geq \ell + 1$

Proof:

Controllability

Lemma: Consider $\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots | AB \mid B] \ \in \mathbb{R}^{n\times\ell}$

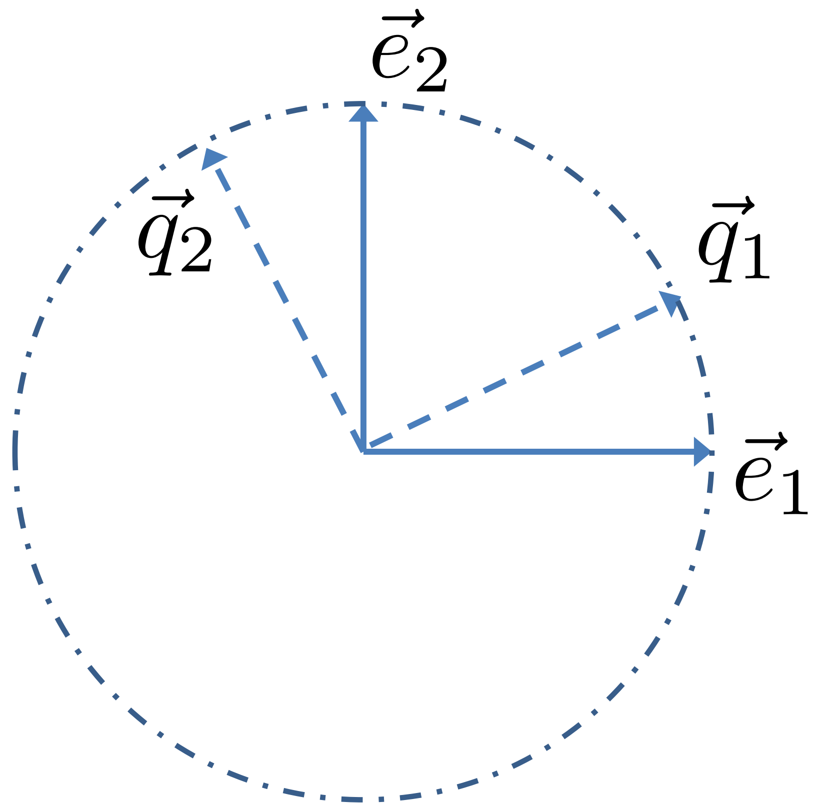
If $\operatorname{rank}[\mathcal{C}_{\ell+1}] = \operatorname{rank}[\mathcal{C}_\ell]$ then $\operatorname{rank}[\mathcal{C}_m] = \operatorname{rank}[\mathcal{C}_\ell]$ for all $m \geq \ell + 1$

Orthonormal Bases and Matrix

Definition: A set of vectors as columns of a matrix $Q = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \in \mathbb{R}^{n \times k}$ are said to be ***orthonormal*** if

$$\vec{q}_i^\top \vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \quad \text{(orthogonal)} \\ 1 & \text{if } i = j \quad \text{(normalized)} \end{cases}$$

Orthonormal Bases and Matrix (Examples)



Orthonormal Bases or Matrix: Properties

Isometric

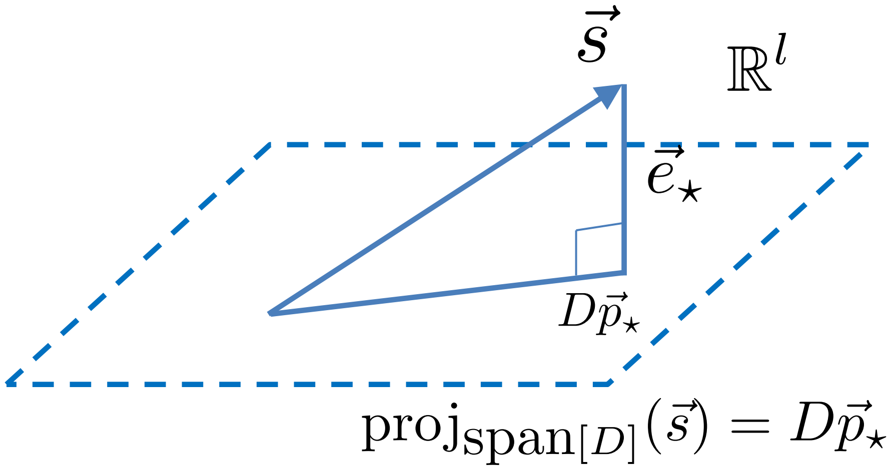
Invertible (and determinante)

Multiplicative

Orthonormal Bases: Projection

Least Squares: $\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2 \quad = (D^\top D)^{-1} D^\top \vec{s}$

If $D = [\vec{d}_1, \vec{d}_2, \dots, \vec{d}_k]$ orthonormal: $D^\top D = I$



Orthonormalization: QR Decomposition

What if columns of $D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k]$ are not orthonormal? Consider the QR decomposition:

$(\text{rank}[D] = k)$

$$[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$$

QR Decomposition & Least Squares

Gram-Schmidt Procedure

Gram-Schmidt via illustration: $D = [\vec{d}_1, \vec{d}_2]$ in $\mathbb{R}^2$

Gram-Schmidt Procedure

Gram-Schmidt via algebraic derivation: $D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k]$ in $\mathbb{R}^n$

Gram-Schmidt Procedure (Summary)

$$\vec{z}_1 = \vec{d}_1 \qquad \vec{q}_1 = \vec{z}_1 / \|\vec{z}_1\|$$

$$\vec{z}_2 = \vec{d}_2 - (\vec{d}_2^\top \vec{q}_1)\vec{q}_1 \qquad \vec{q}_2 = \vec{z}_2 / \|\vec{z}_2\|$$

$$\vec{z}_3 = \vec{d}_3 - (\vec{d}_3^\top \vec{q}_1)\vec{q}_1 - (\vec{d}_3^\top \vec{q}_2)\vec{q}_2 \qquad \vec{q}_3 = \vec{z}_3 / \|\vec{z}_3\|$$

$$\vdots \qquad \vdots$$

$$\vec{z}_k = \vec{d}_k - \sum_{j=1}^{k-1} (\vec{d}_k^\top \vec{q}_j)\vec{q}_j \qquad \vec{q}_k = \vec{z}_k / \|\vec{z}_k\|$$

Claim: 1. $\vec{z}_j^\top \vec{q}_i = 0$ for all $i < j$ 2. $\|\vec{z}_i\| = \vec{d}_i^\top \vec{q}_i$

Gram-Schmidt Procedure & QR

$$\vec{d}_1 = (\vec{d}_1^\top \vec{q}_1)\vec{q}_1$$

$$\vec{d}_2 = (\vec{d}_2^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_2^\top \vec{q}_2)\vec{q}_2$$

$$\vec{d}_3 = (\vec{d}_3^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_3^\top \vec{q}_2)\vec{q}_2 + (\vec{d}_3^\top \vec{q}_3)\vec{q}_3$$

$$\vdots$$

$$\vec{d}_k = (\vec{d}_k^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_k^\top \vec{q}_2)\vec{q}_2 + \cdots + (\vec{d}_k^\top \vec{q}_k)\vec{q}_k$$

$$[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$$

$$(r_{ij} = \vec{d}_j^\top \vec{q}_i)$$