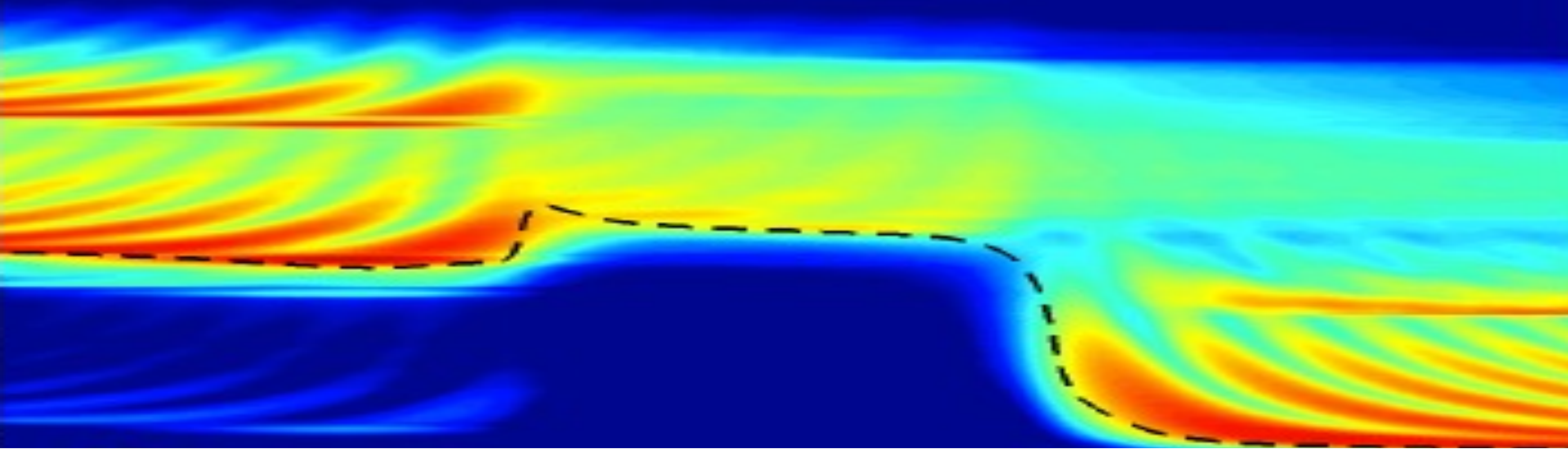


EECS 16B
Designing Information Devices and Systems II
Lecture 17

Prof. Yi Ma

Department of Electrical Engineering and Computer Sciences, UC Berkeley,
yima@eecs.berkeley.edu

Outline

- Condition for Controllability
- Orthonormal Bases and Orthogonal Matrix
- Orthonormalization (Gram-Schmidt procedure)

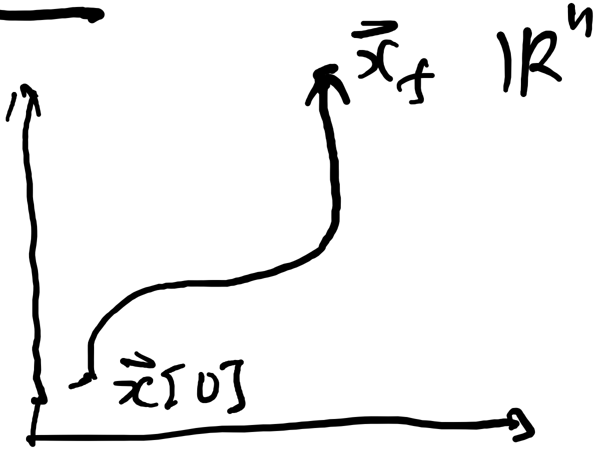
Controllability

Definition: a system $\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ is said to be ***controllable*** if given any target state $\vec{x}_f \in \mathbb{R}^n$ and initial state $\vec{x}[0]$, we can find a time $i = \ell$ and a sequence of control input $u[0], \dots, u[\ell]$ such that $\vec{x}[\ell] = \vec{x}_f$

$$\vec{x}[\ell] = A^{\ell}\vec{x}[0] + A^{\ell-1}Bu[0] + \cdots + ABu[\ell-2] + Bu[\ell-1]$$

$$\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots \mid AB \mid B] \;\in \mathbb{R}^{n\times\ell} \qquad \vec{x}[\ell] = A^{\ell}\vec{x}[0] + \mathcal{C}_\ell \vec{u}[\ell]$$

$$\vec{u}[\ell] = \begin{bmatrix} u[0] \\ \vdots \\ u[\ell-1] \end{bmatrix}$$



Condition for Controllability: $\text{span}[\mathcal{C}_\ell] = \mathbb{R}^n$ or $\text{rank}[\mathcal{C}_\ell] = n$

$$\vec{x}_f = A^{\ell}\vec{x}[0] + \boxed{\mathcal{C}_\ell}\,\vec{u}[\ell]$$

$$\underbrace{\vec{x}_f - A^{\ell}\vec{x}[0]}_{\Delta \in \mathbb{R}^n} \in \underbrace{\text{Col}\{\mathcal{C}_\ell\}}_{\text{span}\{\mathcal{C}_\ell\},\ \text{range}\{\mathcal{C}_\ell\}} = \mathbb{R}^n \iff \text{rank}\{\mathcal{C}_\ell\} = n.$$

$\ell \geq n$ necessary

$A \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $B \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $A^2B \; AB \; B$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightarrow$ $\ell = n$ sufficient?

Controllability

Lemma: Consider $\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots \mid AB \mid B] \in \mathbb{R}^{n\times\ell}$, if $\boxed{\operatorname{rank}[\mathcal{C}_{\ell+1}] = \operatorname{rank}[\mathcal{C}_\ell]}$ then $\operatorname{rank}[\mathcal{C}_m] = \operatorname{rank}[\mathcal{C}_\ell]$ for all $m \geq \ell + 1$

$$C_{\ell+1} = \{A^\ell B, \underbrace{A^{\ell-1}B, \ldots B}_{C_\ell}\}$$

Proof:

$$A^\ell B = a_1 A^{\ell-1} B + a_2 A^{\ell-2} B + \cdots + a_0 B$$

$$C_{\ell+2} = [A^{\ell+1}B, A^\ell B, \underbrace{A^{\ell-1}B, \ldots, B}_{C_\ell}]$$

$$A^{\ell+1}B = A(a_1 A^{\ell-1}B + a_2 A^{\ell-2}B + \cdots + a_0 B)$$

$$= a_1 A^\ell B + a_2 A^{\ell-1} B + \cdots + a_0 AB$$

$$= * A^{\ell-1} B + \cdots + * B.$$

by induction

$\boxed{}$

$$\operatorname{rank} \underbrace{\{A^{n-1}B, \ldots, AB, B\}}_{C_n}$$

controllable iff $\operatorname{rank}(C_n) = n.$

Orthonormal Bases and Orthogonal Matrix

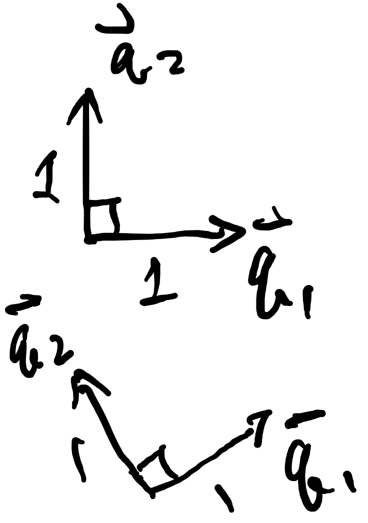
Definition: A set of vectors as columns of a matrix $Q = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \in \mathbb{R}^{n\times k}$ are said to be ***orthonormal*** if

$$\vec{q}_i^\top \vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \quad \text{(orthogonal)} \\ 1 & \text{if } i = j \quad \text{(normalized)} \end{cases}$$

$M_1 M_2$

I ?

$$Q^T Q = \begin{bmatrix} \vec{q}_1^{\,T} \\ \vdots \\ \vec{q}_k^{\,T} \end{bmatrix} [\vec{q}_1, \ldots, \vec{q}_k] = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \ 1 \end{bmatrix}$$



Ce invertible

$T A T^{-1}$ canonical

$Ax = y$

T^{-1}

$k = n$ $\quad Q_{n\times n}$ $\quad$ complete orthonormal bases

$$Q^T Q = I_{n\times n} = Q Q^T \qquad Q^T = Q^{-1}$$

Orthonormal Bases and Matrix (Examples)

$$\vec{q}_1 = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$Q = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \longleftarrow \det(Q) = 1$$

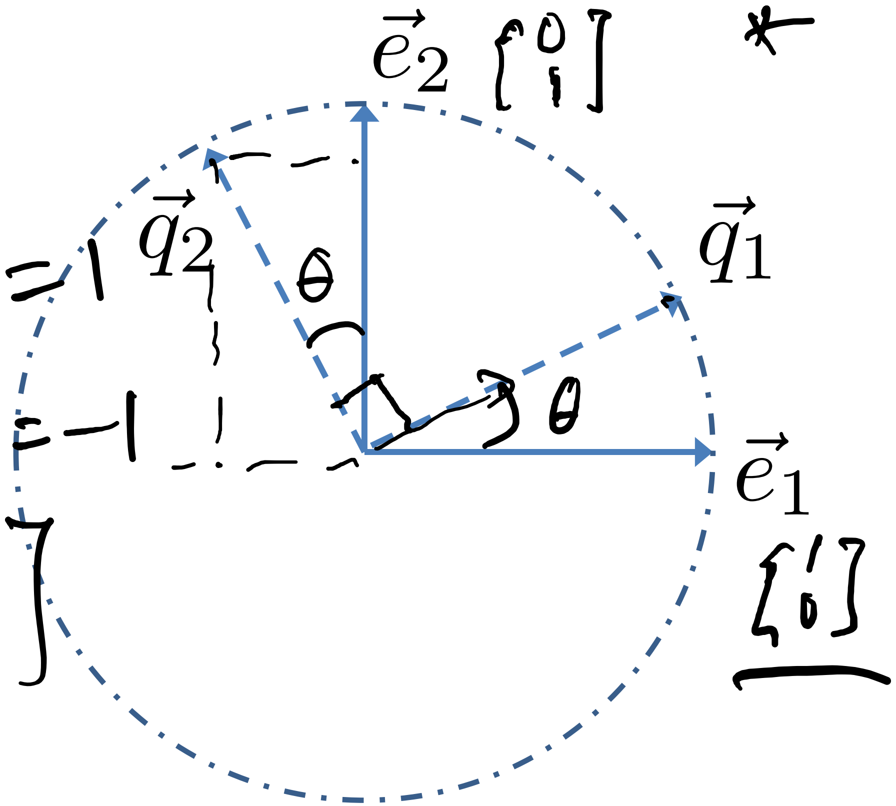
$$\det(Q) = -1 \qquad \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$\vec{q}_i^{\,T}\vec{q}_i = \cos^2\theta + \sin^2\theta = 1$$

$$Q\vec{e}_1 = \underbrace{\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}}_{\vec{q}_1} \qquad Q\vec{e}_2 = \underbrace{\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}}_{\vec{q}_2}$$

$$Q(\theta_1)\,Q(\theta_2) = Q(\theta_1 + \theta_2)$$

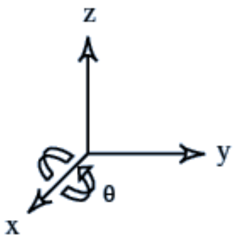
*



Orthonormal Bases and Matrix (Examples)

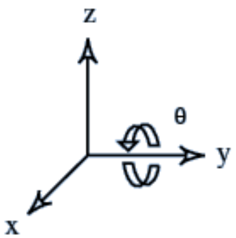
Rotation around the x-Axis

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$



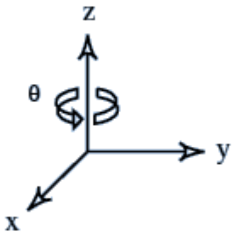
Rotation around the y-Axis

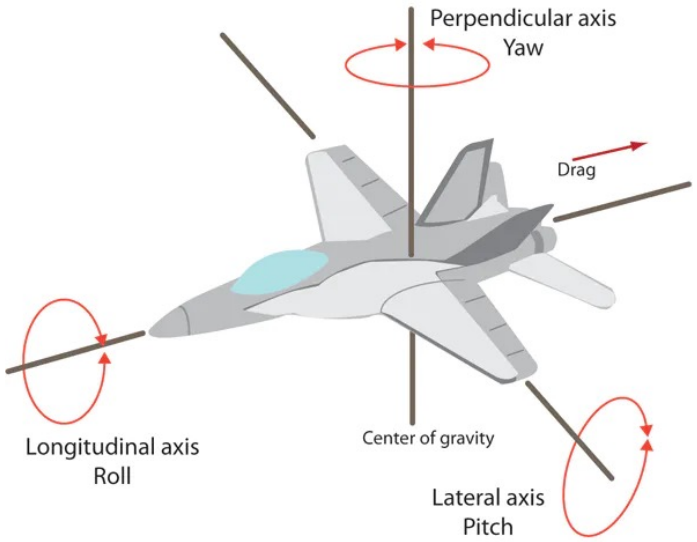
$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$



Rotation around the z-Axis

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$





$$R = R_z(\alpha)\,R_y(\beta)\,R_x(\gamma) = \overbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{yaw}} \overbrace{\begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}}^{\text{pitch}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}}^{\text{roll}}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

Orthonormal Bases or Matrix: Properties

Isometric

$\|Q\vec{x}\|_2 = \|\vec{x}\|_2$

$Q_1 Q_2 Q_3 \cdots\cdots Q_k \vec{x}$

$\|Qx\|_2^2 = (Qx)^T(Qx)$

$= x^T Q^T Q x = x^T x$

Invertible (and determinant) ← square

$QQ^T = Q^TQ = I$

$(Q_1 Q_2)^{-1} = (Q_1 Q_2)^T = Q_2^T Q_1^T$

$\det(QQ^T) = 1 = \det(Q)^2$

$\det(Q) = ?$

± 1

Multiplicative

Q_1, Q_2 - orthogonal

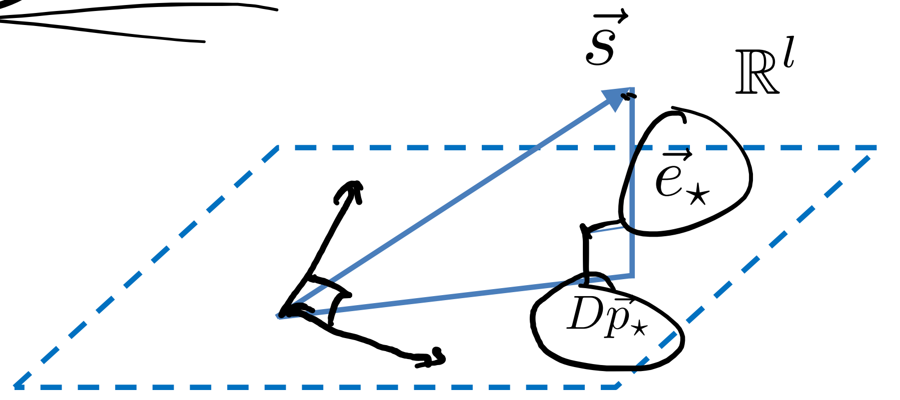
$Q_1 Q_2$ - orthogonal

$Q_1 Q_2 \overset{?}{=} Q_2 Q_1$

Orthonormal Bases: Projection

Least Squares: $\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2 \quad = (D^\top D)^{-1} D^\top \vec{s}$

If $D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k]$ orthonormal: $D^\top D = I$



$\text{proj}_{\text{span}[D]}(\vec{s}) = D\vec{p}_\star$

$$\vec{p}_\star = (D^T D)^{-1} D^T \vec{s} = D^T \vec{s}$$

$$= \begin{bmatrix} \vec{d}_1^T \vec{s} \\ \vec{d}_2^T \vec{s} \\ \vdots \\ \vec{d}_k^T \vec{s} \end{bmatrix}$$

$\text{proj}(\vec{s})$

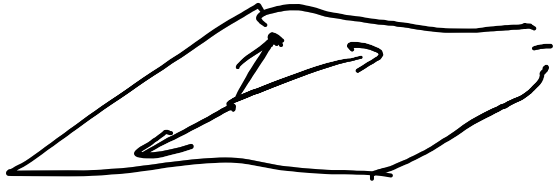
$$D\vec{p}_\star = [\vec{d}_1\ \vec{d}_2\ \cdots\ \vec{d}_k] \begin{bmatrix} \vec{d}_1^T \vec{s} \\ \vec{d}_2^T \vec{s} \\ \vdots \\ \vec{d}_k^T \vec{s} \end{bmatrix} = (\vec{d}_1^T \vec{s})\vec{d}_1 + (\vec{d}_2^T \vec{s})\vec{d}_2 + \cdots + (\vec{d}_k^T \vec{s})\vec{d}_k$$

Orthonormalization: QR Decomposition

What if columns of $D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k]$ are not orthonormal? Consider the QR decomposition:

$(\text{rank}[D] = k)$

$$\underbrace{[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k]}_{D} = \underbrace{[\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k]}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}}_{R}$$



$\text{span}(D) = \text{span}(Q)$

Q orthonormal

upper triangular

$$\underline{y} = \underline{A}x = QRx \qquad \boxed{Q^T y = Rx} = \begin{bmatrix} r_{11} & & * \\ & \ddots & \\ 0 & & r_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$

QR Decomposition & Least Squares

Least Squares: $\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2 \quad = (D^\top D)^{-1} D^\top \vec{s}$

$D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = QR$

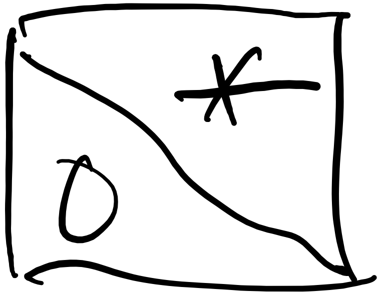
$$D^T \underbrace{(\vec{s} - D\vec{p}_*)}_{\vec{e}_*} = 0$$

$$D^T \vec{s} = D^T D \vec{p}$$

$$R^T Q^T \vec{s} = R^T Q^T Q R \vec{p}$$

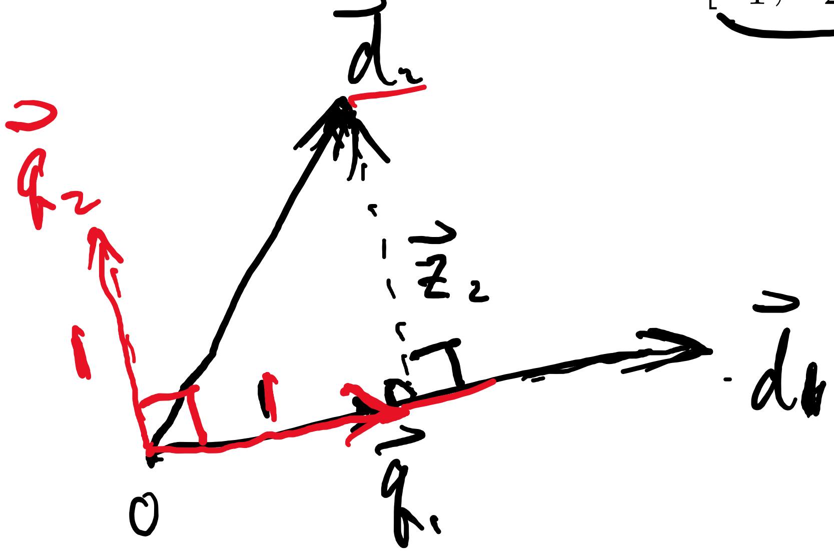
$$R^T Q^T \vec{s} = R^T R \vec{p}$$

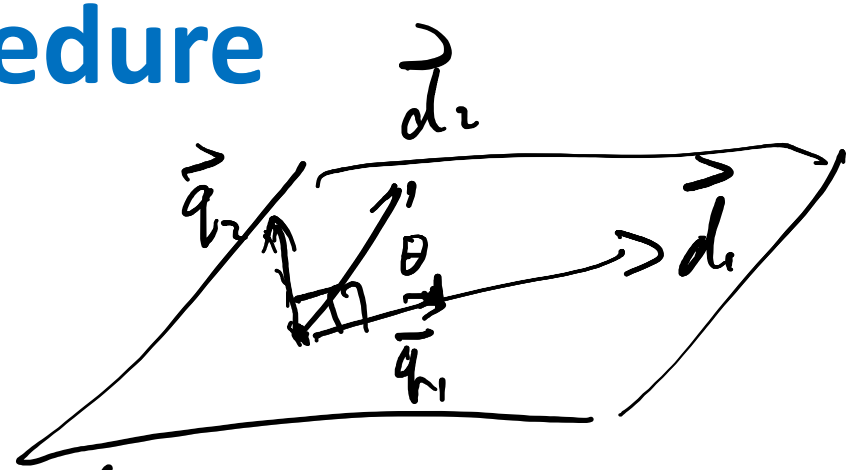
$$Q^T \vec{s} = R \vec{p}$$



Gram-Schmidt Procedure

Gram-Schmidt via illustration: $D = [\vec{d}_1, \vec{d}_2]$ in $\mathbb{R}^n$





1. $\vec{q}_1 = \vec{d}_1 / \|\vec{d}_1\|_2$

2. $\vec{q}_2$? project $\vec{d}_2$ onto span$(\vec{q}_1)$

compute residual

$$\vec{z}_2 = \vec{d}_2 - (\vec{d}_2^T \vec{q}_1)\, \vec{q}_1$$

$$\vec{q}_2 = \vec{z}_2 / \|z_2\|_2$$

Gram-Schmidt Procedure

Gram-Schmidt via algebraic derivation: $D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k]$ in $\mathbb{R}^n$

$\vec{d}_3$ $\quad$ $\vec{q}_3$? project $\vec{d}_3$ onto span$(\vec{q}_1, \vec{q}_2)$

compute the residual

$$\underline{\vec{z}_3} = \vec{d}_3 - \underline{\left((\vec{d}_3^T \vec{q}_1)\vec{q}_1 + (\vec{d}_3^T \vec{q}_2)\vec{q}_2 \right)}$$

$$\vec{q}_3 = \vec{z}_3 / \|\vec{z}_3\|$$

Gram-Schmidt Procedure (Summary)

$$\vec{z}_1 = \vec{d}_1 \qquad \vec{q}_1 = \vec{z}_1 / \|\vec{z}_1\|$$

$$\vec{z}_2 = \vec{d}_2 - (\vec{d}_2^\top \vec{q}_1)\vec{q}_1 \qquad \vec{q}_2 = \vec{z}_2 / \|\vec{z}_2\|$$

$$\vec{z}_3 = \vec{d}_3 - (\vec{d}_3^\top \vec{q}_1)\vec{q}_1 - (\vec{d}_3^\top \vec{q}_2)\vec{q}_2 \qquad \vec{q}_3 = \vec{z}_3 / \|\vec{z}_3\|$$

$$\vdots \qquad \vdots$$

$$\vec{z}_k = \vec{d}_k - \sum_{j=1}^{k-1} (\vec{d}_k^\top \vec{q}_j)\vec{q}_j \qquad \vec{q}_k = \vec{z}_k / \|\vec{z}_k\|$$

Claim: 1. $\vec{z}_j^\top \vec{q}_i = 0$ for all $i < j$ 2. $\|\vec{z}_i\| = \vec{d}_i^\top \vec{q}_i$

Gram-Schmidt & QR Decomposition

$$\begin{aligned}
\vec{d}_1 &= (\vec{d}_1^\top \vec{q}_1)\vec{q}_1 \\
\vec{d}_2 &= (\vec{d}_2^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_2^\top \vec{q}_2)\vec{q}_2 \\
\vec{d}_3 &= (\vec{d}_3^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_3^\top \vec{q}_2)\vec{q}_2 + (\vec{d}_3^\top \vec{q}_3)\vec{q}_3 \\
&\vdots \\
\vec{d}_k &= (\vec{d}_k^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_k^\top \vec{q}_2)\vec{q}_2 + \cdots + (\vec{d}_k^\top \vec{q}_k)\vec{q}_k
\end{aligned}$$

$$(r_{ij} = \vec{d}_j^\top \vec{q}_i)$$

$$[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$$