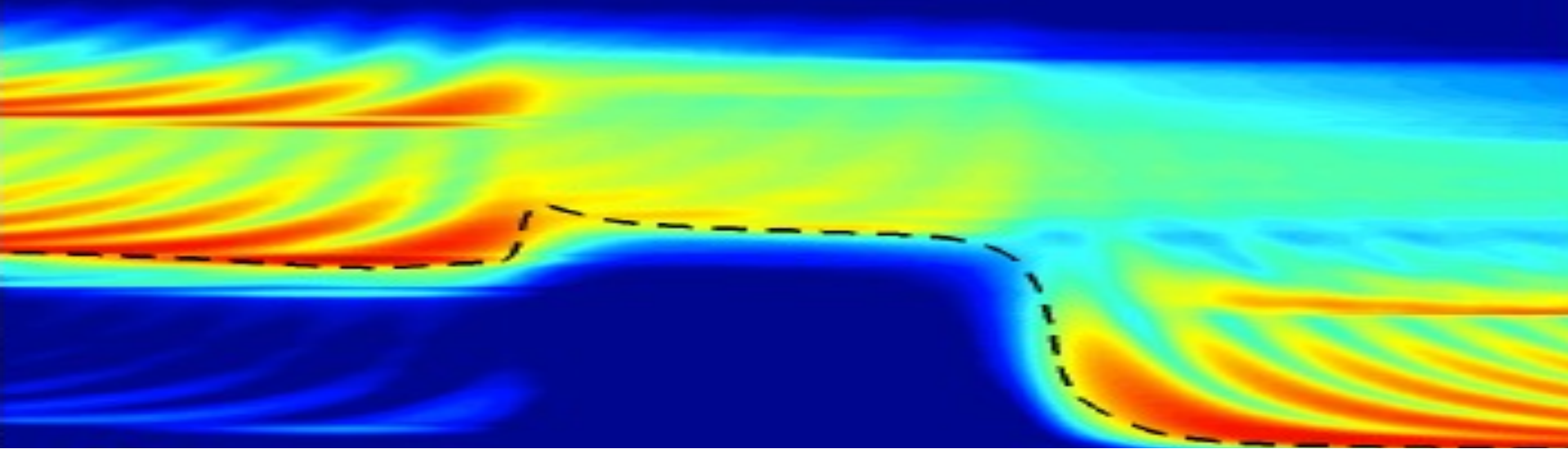

EECS 16B
Designing Information Devices and Systems II
Lecture 18

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Outline

- Orthonormalization (Gram-Schmidt) and QR Decomposition
- Upper Triangularization

Gram-Schmidt Procedure (Summary)

$$\vec{z}_1 = \vec{d}_1 \qquad \vec{q}_1 = \vec{z}_1 / \|\vec{z}_1\|$$

$$\vec{z}_2 = \vec{d}_2 - (\vec{d}_2^\top \vec{q}_1)\vec{q}_1 \qquad \vec{q}_2 = \vec{z}_2 / \|\vec{z}_2\|$$

$$\vec{z}_3 = \vec{d}_3 - (\vec{d}_3^\top \vec{q}_1)\vec{q}_1 - (\vec{d}_3^\top \vec{q}_2)\vec{q}_2 \qquad \vec{q}_3 = \vec{z}_3 / \|\vec{z}_3\|$$

$$\vdots \qquad \vdots$$

$$\vec{z}_k = \vec{d}_k - \sum_{j=1}^{k-1} (\vec{d}_k^\top \vec{q}_j)\vec{q}_j \qquad \vec{q}_k = \vec{z}_k / \|\vec{z}_k\|$$

Claim: 1. $\vec{z}_j^\top \vec{q}_i = 0$ for all $i < j$ 2. $\|\vec{z}_i\| = \vec{d}_i^\top \vec{q}_i$

Gram-Schmidt & QR Decomposition

$$\vec{d}_1 = (\vec{d}_1^\top \vec{q}_1)\vec{q}_1$$

$$\vec{d}_2 = (\vec{d}_2^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_2^\top \vec{q}_2)\vec{q}_2$$

$$\vec{d}_3 = (\vec{d}_3^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_3^\top \vec{q}_2)\vec{q}_2 + (\vec{d}_3^\top \vec{q}_3)\vec{q}_3$$

$$\vdots$$

$$\vec{d}_k = (\vec{d}_k^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_k^\top \vec{q}_2)\vec{q}_2 + \cdots + (\vec{d}_k^\top \vec{q}_k)\vec{q}_k$$

$$(r_{ij} = \vec{d}_j^\top \vec{q}_i)$$

$$[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$$

QR, Diagonalization, Triangularization

QR Decomposition for $D \in \mathbb{R}^{n \times k}$: $[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$

Diagonalization for $A \in \mathbb{R}^{n \times n}$: $A[\vec{v}_1, \ldots, \vec{v}_n] = [\lambda_1 \vec{v}_1, \ldots, \lambda \vec{v}_n] = [\vec{v}_1, \ldots, \vec{v}_n] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$

Triangularization for $A \in \mathbb{R}^{n \times n}$: $A[\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n] = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{bmatrix}$

Diagonalization v.s. Triangularization

Conditions for diagonalization of $A \in \mathbb{R}^{n\times n}$: $V^{-1}AV = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$

Upper-Triangularization

Upper-triangularization for $A \in \mathbb{R}^{n\times n}$: $U^{-1}AU = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{bmatrix}$

Eigenvalues of an upper-triangular matrix:

Upper-Triangularization

Solution to an upper-triangular system of linear equations:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Upper-Triangularization

Solution to an upper-triangular system of linear differential equations:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(t)$$

Upper-Triangularization (Schur Decomposition)

Claim: For any matrix $A \in \mathbb{R}^{n \times n}$ with real eigenvalues, there exists an orthogonal matrix: $U \in \mathbb{R}^{n \times n}$ such that $U^\top U = I$ and

$$R = U^{-1}AU = U^\top AU = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{bmatrix}$$

Proof:

Upper-Triangularization (Schur Decomposition)

Claim: For any matrix $A \in \mathbb{R}^{n \times n}$ with real eigenvalues, there exists an orthogonal matrix: $U \in \mathbb{R}^{n \times n}$ such that $U^\top U = I$ and $R = U^{-1}AU = U^\top AU$ is upper-triangular.

Proof (continued):

Upper-Triangularization (Schur Decomposition)

Claim: For any matrix $A \in \mathbb{R}^{n \times n}$ with real eigenvalues, there exists an orthogonal matrix: $U \in \mathbb{R}^{n \times n}$ such that $U^\top U = I$ and $R = U^{-1}AU = U^\top AU$ is upper-triangular.

Proof (continued):