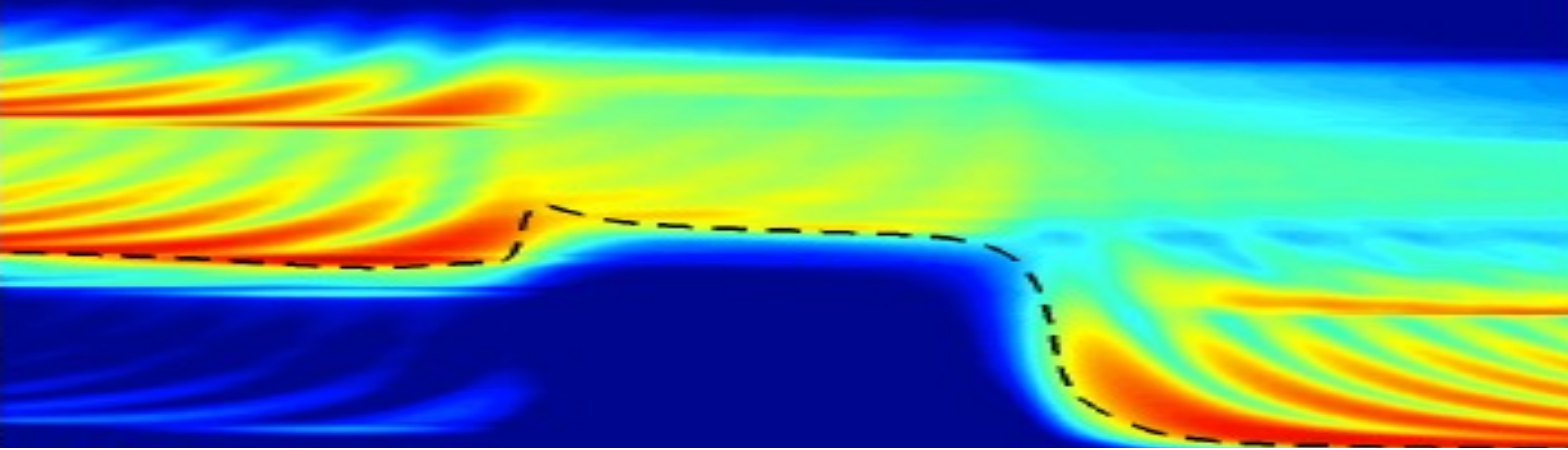

EECS 16B
Designing Information Devices and Systems II
Lecture 18

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Outline

- Orthonormalization (Gram-Schmidt) and QR Decomposition
- Upper Triangularization

Gram-Schmidt Procedure (Summary)

$$D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k]_{n\times k}$$

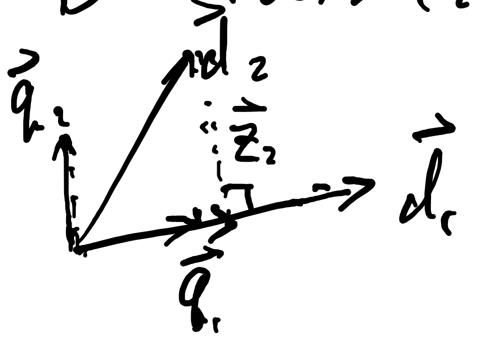
$$\vec{z}_1 = \vec{d}_1 \qquad \vec{q}_1 = \vec{z}_1/\|\vec{z}_1\|$$

$$\vec{z}_2 = \vec{d}_2 - (\vec{d}_2^\top \vec{q}_1)\vec{q}_1 \qquad \vec{q}_2 = \vec{z}_2/\|\vec{z}_2\|$$

$$\vec{z}_3 = \vec{d}_3 - (\vec{d}_3^\top \vec{q}_1)\vec{q}_1 - (\vec{d}_3^\top \vec{q}_2)\vec{q}_2 \qquad \vec{q}_3 = \vec{z}_3/\|\vec{z}_3\|$$

$$\vdots \qquad \vdots$$

$$\boxed{\vec{z}_k = \vec{d}_k - \sum_{j=1}^{k-1}(\vec{d}_k^\top \vec{q}_j)\vec{q}_j} \qquad \vec{q}_k = \vec{z}_k/\|\vec{z}_k\|$$



$$\vec{q}_\ell^{\,\top} \vec{q}_j = \begin{cases} 0 & \ell \neq j \\ 1 & \ell = j \end{cases}$$

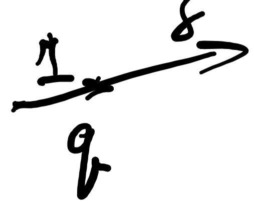
Claim: 1. $\vec{z}_j^\top \vec{q}_i = 0$ for all $i < j$ 2. $\|\vec{z}_i\| = \vec{d}_i^\top \vec{q}_i$

① $\vec{z}_j^\top \vec{q}_i = \left(\vec{d}_j - \sum_{\ell=1}^{j-1} (\vec{d}_j^\top \vec{q}_\ell)\vec{q}_\ell\right)^\top \vec{q}_i$

$= \vec{d}_j^\top \vec{q}_i - (\vec{d}_j^\top \vec{q}_i) = 0$

② $\|\vec{z}_j\|$

$\vec{z}_j = \|\vec{z}_j\| \vec{q}_j$



$\vec{z}_j \cdot \vec{q}_j = \vec{d}_j^\top \vec{q}_j - 0$

$\|\vec{z}_j\| = \vec{d}_j^\top \vec{q}_j$

Gram-Schmidt & QR Decomposition

$Q_{n\times k}^T x = 0$

$$\vec{d}_1 = (\vec{d}_1^\top \vec{q}_1)\vec{q}_1$$
$$\vec{d}_2 = (\vec{d}_2^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_2^\top \vec{q}_2)\vec{q}_2$$
$$\vec{d}_3 = (\vec{d}_3^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_3^\top \vec{q}_2)\vec{q}_2 + (\vec{d}_3^\top \vec{q}_3)\vec{q}_3$$
$$\vdots$$
$$\vec{d}_k = (\vec{d}_k^\top \vec{q}_1)\vec{q}_1 + (\vec{d}_k^\top \vec{q}_2)\vec{q}_2 + \cdots + \underbrace{(\vec{d}_k^\top \vec{q}_k)}_{\|z_k\|}\underbrace{\vec{q}_k}_{q_k}$$

$$(r_{ij} = \vec{d}_j^\top \vec{q}_i)$$

$$[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$$

$$D_{n\times k} = Q_{n\times k} R_{k\times k}$$

R

$$D_{n\times n} = Q_{n\times n} R_{n\times n}$$

$Q^T Q = I = Q Q^T$ — orthogonal

complete

$$D_{n\times k} = Q_{n\times k} R_{k\times k}$$

$$Q = [Q_{n\times k}, \tilde{Q}_{(n-k)\times k}]$$

$\tilde{Q}^T Q_{n\times k} = 0$

QR, Diagonalization, Triangularization

QR Decomposition for $D \in \mathbb{R}^{n\times k}$: $[\vec{d}_1, \vec{d}_2, \dots, \vec{d}_k] = [\vec{q}_1, \vec{q}_2, \dots, \vec{q}_k] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$

$y = Ax$

$\boxed{D = QR}$ $\boxed{Q^T D = R}$

Diagonalization for $A \in \mathbb{R}^{n\times n}$: $\underbrace{A[\vec{v}_1, \dots, \vec{v}_n] = [\lambda_1 \vec{v}_1, \dots, \lambda \vec{v}_n]}_{V} = \underbrace{[\vec{v}_1, \dots, \vec{v}_n]}_{V} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}}_{\Lambda}$

$AV = V\Lambda$

$\boxed{V^{-1}AV = \Lambda}$

Triangularization for $A \in \mathbb{R}^{n\times n}$: $A\underbrace{[\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n]}_{U} = \underbrace{[\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n]}_{U} \underbrace{\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}}_{T}$

$AU = UT$

$\boxed{U^{-1}AU = T}$

U - orthogonal

$U^T A U = T$

Diagonalization v.s. Triangularization

Conditions for diagonalization of $A \in \mathbb{R}^{n\times n}$: $\quad V^{-1}AV = \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}}_{\Lambda}$

$AV = V\Lambda$

$Av_i = \lambda_i v_i \quad i = 1, \dots, n.$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

V^{-1}

$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

Jordan

$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ & \lambda & 1 & 0 & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}$ A

$(A - \lambda I)v = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{bmatrix}_{n\times n} v = 0$

$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$Av = \lambda v$

Upper-Triangularization

$-p^n(\lambda) = 0$

$\cancel{\det(\lambda I - A)}$

Upper-triangularization for $A \in \mathbb{R}^{n\times n}$: $U^{-1}AU = \underbrace{\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}}_{T}$

$y = U^{-1}x$

$x[i+1] = A\, x[i]$

$y[i+1] = \underbrace{U^{-1}AU}_{\tilde{A}}\, y[i]$

Eigenvalues of an upper-triangular matrix:

$\det(\lambda I - \tilde{A}) = \underbrace{\det(U)}_{\alpha} \det(\lambda I - \tilde{A}) \underbrace{\det(U^{-1})}_{\alpha^{-1}}$

$= \det(U\lambda I U^{-1} - U\tilde{A}U^{-1})$

$= \det(\lambda I - A)$

$\det(\lambda I - T) = \det \begin{bmatrix} \lambda - t_{11} & & * \\ & \lambda - t_{22} & \\ 0 & & \ddots & \\ & & & \lambda - t_{nn} \end{bmatrix} = \boxed{(\lambda - t_{11}) \cdots (\lambda - t_{nn})}$

Upper-Triangularization

Solution to an upper-triangular system of linear equations:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$y = Ax$

$x = A^{-1}y$

$y_n = t_{nn}x_n \leftarrow x_n = t_{nn}^{-1} y_n$

$y_{n-1} = t_{n-1,n-1}x_{n-1} + \underline{t_{n-1,n}x_n} \leftarrow \underline{x_{n-1}}$

$\vdots$

$y_1 = t_{11}x_1 + \underline{\text{known}}$

Upper-Triangularization

Solution to an upper-triangular system of linear differential equations:

$$\dot{x}(t) = A\,x(t) \qquad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(t)$$

$$A = QR$$

$$Q^T \dot{x}(t) = R\,x(t)$$

$$\frac{dx_n(t)}{dt} = t_{nn}\,x_n(t) + b_n u(t)$$

$$x_n(t) = \underbrace{e^{t_{nn}(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{t_{nn}(\tau - t_0)} b_n u(\tau)\,d\tau}$$

$$\frac{dx_{n-1}(t)}{dt} = \underline{t_{n-1,n-1}}\,x_{n-1}(t) + \underbrace{t_{n-1,t}\,x_n(t) + b_{n-1}u(t)}_{\tilde{u}_{n-1}(t)}$$

$$\vdots$$

$$\frac{dx_1(t)}{dt} = t_{11}\,x_1(t) + \underline{\text{known}}$$

Upper-Triangularization (Schur Decomposition)

Claim: For any matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues, there exists an orthogonal matrix: $U \in \mathbb{R}^{n\times n}$ such that $U^\top U = I$ and

$$T = U^{-1}AU = U^\top AU = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

T^*

$\dot{x}(t) = Ax(t)$

$z(t) = U^{-1}x(t)$

$x(t) = Uz(t)$

$z(t) = U^T x(t)$

Proof (by induction):

1\. $n=1$ $A = (a_{11})$ true

2\. $n=k+1$ $A_{(k+1)\times(k+1)}$ can be triangulated

if $\forall A_{k\times k}$ can be triangulated

$A \in \mathbb{R}^{(k+1)\times(k+1)}$

$AU = UT$ $\begin{bmatrix} * & \\ 0 & * \end{bmatrix}$

$A\vec{u}_1 = t_{11}\vec{u}_1$

Upper-Triangularization (Schur Decomposition)

Claim: For any matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues, there exists an orthogonal matrix: $U \in \mathbb{R}^{n\times n}$ such that $U^\top U = I$ and $T = U^{-1}AU = U^\top AU$ is upper-triangular.

normali

Proof (continued):

$\vec{q}_1$ to be the eigenvector w.r.t λ_1 of A_{k+1} $\quad A\vec{q}_1 = \lambda_1 \vec{q}_1$

$Q = [\vec{q}_1\ \underbrace{\vec{q}_2, \ldots, \vec{q}_{k+1}}_{G.S.}]$ — orthogonal $\quad Q^TQ = I$

$$Q^TAQ = \begin{bmatrix} \vec{q}_1^{\,T} \\ \vec{q}_2^{\,T} \\ \vdots \\ \vec{q}_{k+1}^{\,T} \end{bmatrix} [\lambda_1\vec{q}_1\ A\vec{q}_2, \ldots A\vec{q}_{k+1}] = \begin{bmatrix} \lambda_1 & \tilde{a}_{12} \\ 0 & \\ \vdots & \tilde{A}_{22} \\ 0 & \end{bmatrix}$$

$\lambda =$, λ, $k \times k$

$\tilde{A}_{k+1}$

Upper-Triangularization (Schur Decomposition)

Claim: For any matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues, there exists an orthogonal matrix: $U \in \mathbb{R}^{n\times n}$ such that $U^\top U = I$ and $T = U^{-1}AU = U^\top AU$ is upper-triangular.

Proof (continued):

$\tilde{A}_{22}$ $k \times k$ has real eigenvalues

$\det(\lambda I - A) = \det(\lambda I - \tilde{A})$

$= (\lambda - \lambda_1) \det(\lambda I - A_{22})$

Upper-Triangularization (Algorithm)

Algorithm 10 Real Schur Decomposition

Input: A square matrix $A \in \mathbb{R}^{n \times n}$ with real eigenvalues.

Output: An orthonormal matrix $U \in \mathbb{R}^{n \times n}$ and an upper-triangular matrix $T \in \mathbb{R}^{n \times n}$ such that $A = UTU^\top$.

function $\textsc{RealSchurDecomposition}(A)$
    if A is 1×1 **then**
        return $\begin{bmatrix} 1 \end{bmatrix}, A$
    end if
    $(\vec{q}_1, \lambda_1) := \textsc{FindEigenvectorEigenvalue}(A)$
    $Q := \textsc{ExtendBasis}(\{\vec{q}_1\}, \mathbb{R}^n)$ ▷ Extend $\{\vec{q}_1\}$ to a basis of $\mathbb{R}^n$ using Gram-Schmidt; see Note 13
    Unpack $Q := \begin{bmatrix} \vec{q}_1 & \widetilde{Q} \end{bmatrix}$
    Compute and unpack $Q^\top A Q = \begin{bmatrix} \lambda_1 & \vec{\tilde{a}}_{12}^\top \\ \vec{0}_{n-1} & \widetilde{A}_{22} \end{bmatrix}$
    $(P, \widetilde{T}) := \textsc{RealSchurDecomposition}(\widetilde{A}_{22})$
    $U := \begin{bmatrix} \vec{q}_1 & \widetilde{Q}P \end{bmatrix}$
    $T := \begin{bmatrix} \lambda_1 & \vec{\tilde{a}}_{12}^\top P \\ \vec{0}_{n-1} & \widetilde{T} \end{bmatrix}$
    return (U, T)
end function