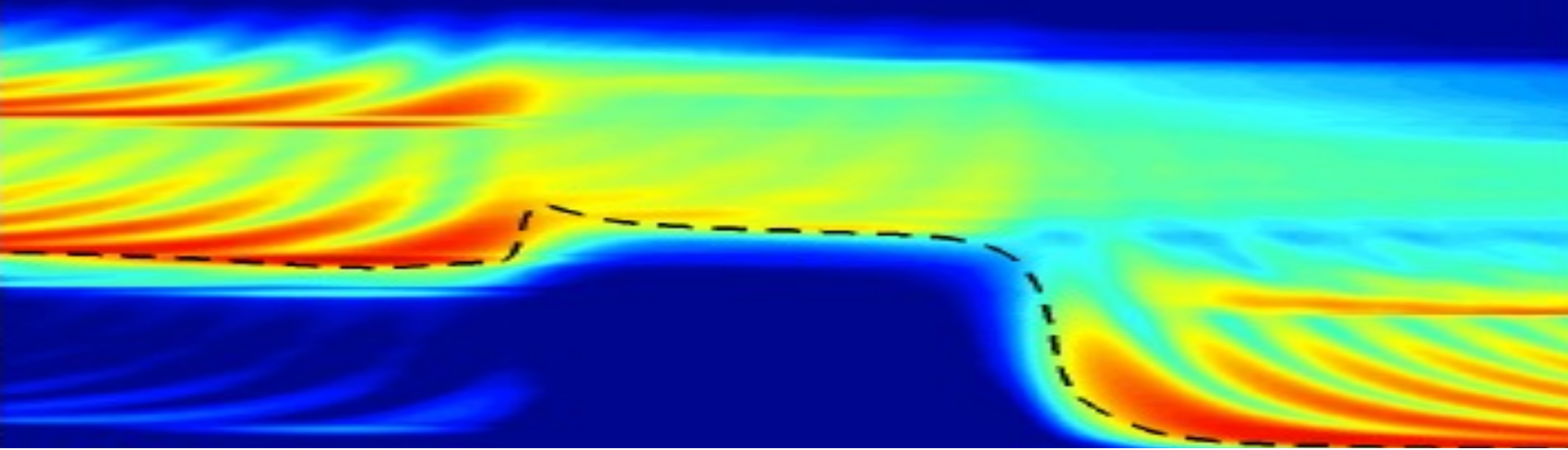

EECS 16B
Designing Information Devices and Systems II
Lecture 19

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Outline

- Upper Triangularization
- The RLC circuit
- Spectral Theorem

Upper-Triangularization (Schur Decomposition)

Claim: For any matrix $A \in \mathbb{R}^{n \times n}$ with real eigenvalues, there exists an orthogonal matrix: $U \in \mathbb{R}^{n \times n}$ such that $U^\top U = I$ and $T = U^{-1}AU = U^\top AU$ is upper-triangular.

Proof (continued):

Upper-Triangularization (Schur Decomposition)

Claim: For any matrix $A \in \mathbb{R}^{n \times n}$ with real eigenvalues, there exists an orthogonal matrix: $U \in \mathbb{R}^{n \times n}$ such that $U^\top U = I$ and $T = U^{-1}AU = U^\top AU$ is upper-triangular.

Proof (continued):

Upper-Triangularization (Algorithm)

Algorithm 10 Real Schur Decomposition

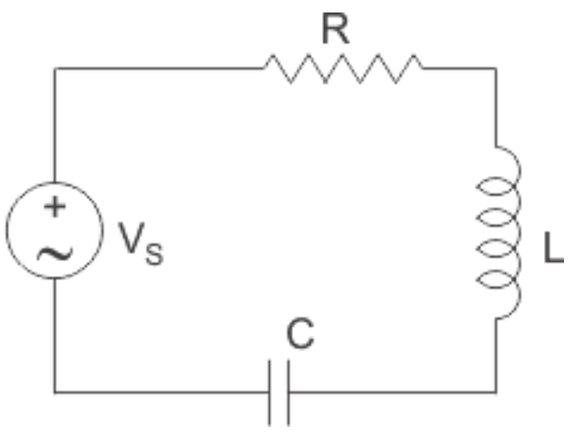
Input: A square matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues.

Output: An orthonormal matrix $U \in \mathbb{R}^{n\times n}$ and an upper-triangular matrix $T \in \mathbb{R}^{n\times n}$ such that $A = UTU^\top$.

function RealSchurDecomposition(A)
    if A is 1×1 **then**
        return $\begin{bmatrix} 1 \end{bmatrix}, A$
    end if
    $(\vec{q}_1, \lambda_1) :=$ FindEigenvectorEigenvalue(A)
    $Q :=$ ExtendBasis($\{\vec{q}_1\}, \mathbb{R}^n$)   ▷ Extend $\{\vec{q}_1\}$ to a basis of $\mathbb{R}^n$ using Gram-Schmidt; see Note 13
    Unpack $Q := \begin{bmatrix} \vec{q}_1 & \widetilde{Q} \end{bmatrix}$
    Compute and unpack $Q^\top A Q = \begin{bmatrix} \lambda_1 & \vec{\tilde{a}}_{12}^\top \\ \vec{0}_{n-1} & \widetilde{A}_{22} \end{bmatrix}$
    $(P, \widetilde{T}) :=$ RealSchurDecomposition($\widetilde{A}_{22}$)
    $U := \begin{bmatrix} \vec{q}_1 & \widetilde{Q}P \end{bmatrix}$
    $T := \begin{bmatrix} \lambda_1 & \vec{\tilde{a}}_{12}^\top P \\ \vec{0}_{n-1} & \widetilde{T} \end{bmatrix}$
    return (U, T)
end function

Upper-Triangularization (Example)

A RLC Circuit



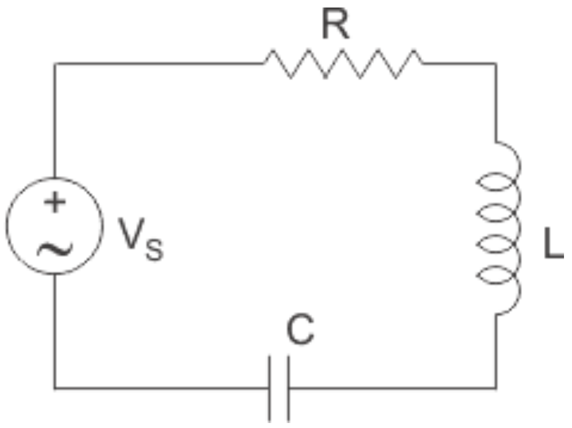
$$i(t) = C\frac{dV_c(t)}{dt}, \quad V_L(t) = L\frac{di(t)}{dt}$$

$$V_s(t) = V_R(t) + V_L(t) + V_c(t)$$

Stability, Controllability
Diagonalization, Triangularization

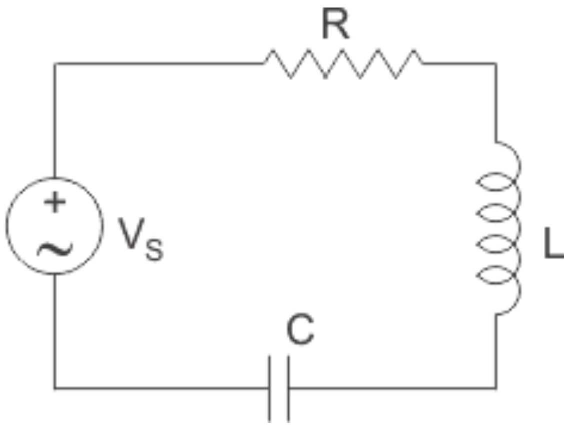
Upper-Triangularization (Example)

A RLC Circuit (critically damped)



Upper-Triangularization (Example)

A RLC Circuit



Spectral Theorem (motivations)

Diagonalization for $A \in \mathbb{R}^{n\times n}$ with n independent eigenvectors: $V^{-1}AV = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$

Triangularization for $A \in \mathbb{R}^{n\times n}$ with real eigenvalues: $U^{-1}AU = U^\top AU = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$

For real symmetric matrices $A = A^\top \in \mathbb{R}^{n\times n}$:

$$V^{-1}AV = V^\top AV = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

Spectral Theorem (statement)

Theorem: Let $A = A^\top \in \mathbb{R}^{n \times n}$ be a *real and symmetric* matrix. Then

1. All eigenvalues of A are real.
2. A is diagonalizable.
3. All eigenvectors are orthogonal to each other.

Spectral Theorem (proof)

Spectral Theorem (extensions)

Consider: $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) \quad \text{with } A \text{ symmetric, and } \lambda_{\max}(A) < -\lambda.$

How does the "energy" $V(t) = \|\vec{x}(t)\|_2^2 = \vec{x}(t)^\top \vec{x}(t)$ evolves?

Spectral Theorem (extensions)

What if A is real and *anti-symmetric*: $A^\top = -A \in \mathbb{R}^{n\times n}$