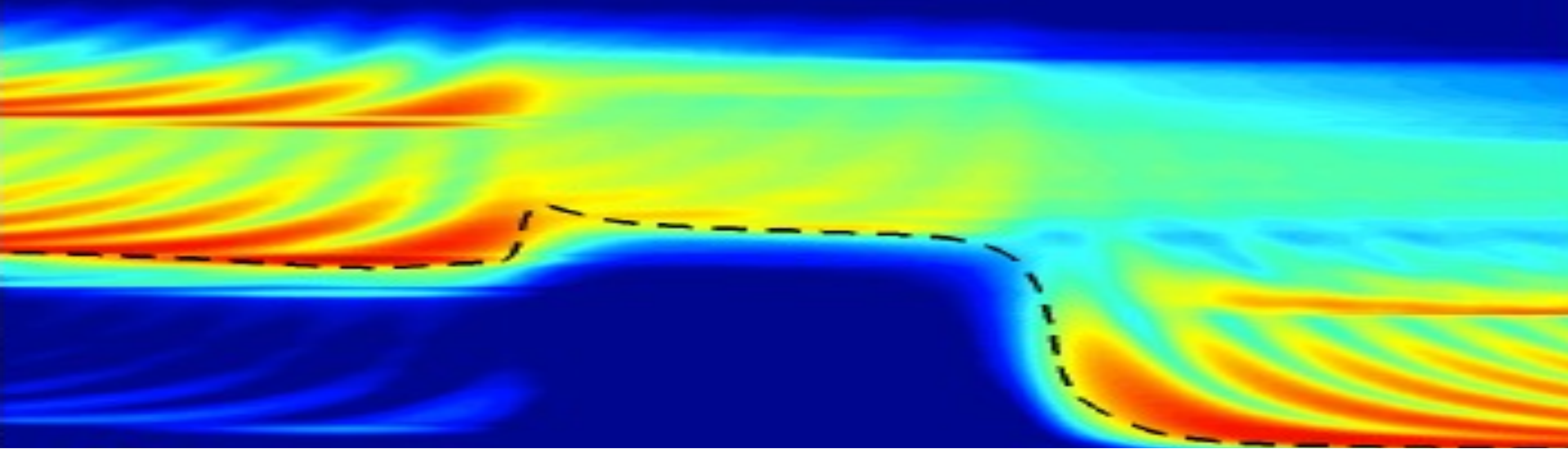


EECS 16B
Designing Information Devices and Systems II
Lecture 19

Prof. Yi Ma
Department of Electrical Engineering and Computer Sciences, UC Berkeley,
yima@eecs.berkeley.edu

Outline

- Upper Triangularization
- An RLC Circuit Example
- Spectral Theorem

Upper-Triangularization (Schur Decomposition)

Claim: For any matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues, there exists an orthogonal matrix: $U \in \mathbb{R}^{n\times n}$ such that $U^\top U = I$ and $T = U^{-1}AU = U^\top AU$ is upper-triangular.

$$\underbrace{A[\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n]}_{U} = \underbrace{[\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n]}_{U} \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

Proof (continued): $n=1$ $k\times k$ is true

want to show true for $(k+1)\times(k+1)$ A.

$A_{k+1} U = U T_{k+1}$ $\quad A\vec{u}_1 = t_{11}\vec{u}_1$

$\lambda_1, \vec{q}_1$ the eigenvalue-eigenvector for A_{k+1}

Q_{k+1} - othogonal $\quad Q = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_{k+1}] \in \mathbb{R}^{(k+1)\times(k+1)}$

$\mathbb{R}^{k+1}$

$Q^T Q = I$ $\quad Q^T A Q = \begin{bmatrix} \vec{q}_1^{\,T} \\ \vec{q}_2^{\,T} \\ \vdots \\ \vec{q}_{k+1}^{\,T} \end{bmatrix} [\underbrace{\lambda_1 \vec{q}_1}, \underbrace{A\vec{q}_2, \ldots, A\vec{q}_{k+1}}]$

Upper-Triangularization (Schur Decomposition)

Claim: For any matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues, there exists an orthogonal matrix: $U \in \mathbb{R}^{n\times n}$ such that $U^\top U = I$ and $T = U^{-1}AU = U^\top AU$ is upper-triangular.

Proof (continued):

$$Q^T A Q = \begin{bmatrix} \lambda_1 & \tilde{a}_{12} \\ 0 & \\ \vdots & \boxed{\tilde{A}_{22}} \\ 0 & \end{bmatrix}$$

$k \times k$

$\tilde{A}_{22}$ $k\times k$ — triangulatable

P_k - orthogonal $P^T P = I$

$P^T \tilde{A}_{22} P = \tilde{T}_{k\times k}$ upper-triang.

$$U = Q \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} \leftarrow (k+1)\times(k+1)$$

$$U^T U = I_{(k+1)\times(k+1)}$$

$$U^T A U = \begin{bmatrix} 1 & 0 \\ 0 & P^T \end{bmatrix} Q^T A Q \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & P^T \end{bmatrix} \begin{bmatrix} \lambda_1 & \tilde{a}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} = \begin{bmatrix} \lambda_1 & \tilde{a}_{12} P \\ 0 & P^T \tilde{A}_{22} P \end{bmatrix}$$

Upper-Triangularization (Algorithm)

Algorithm 10 Real Schur Decomposition

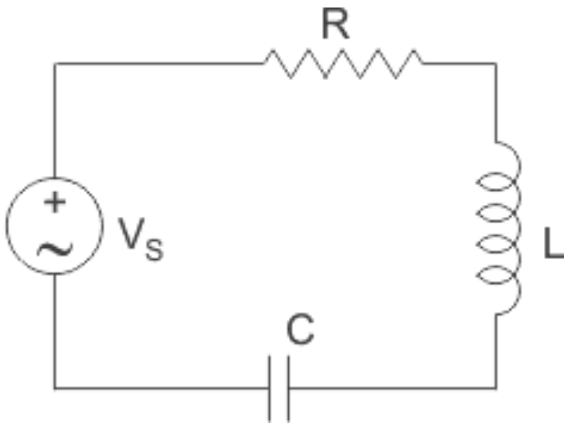
Input: A square matrix $A \in \mathbb{R}^{n \times n}$ with real eigenvalues.

Output: An orthonormal matrix $U \in \mathbb{R}^{n \times n}$ and an upper-triangular matrix $T \in \mathbb{R}^{n \times n}$ such that $A = UTU^\top$.

function $\textsc{RealSchurDecomposition}(A)$
    if A is 1×1 **then**
        return $\begin{bmatrix} 1 \end{bmatrix}, A$
    end if
    $(\vec{q}_1, \lambda_1) := \textsc{FindEigenvectorEigenvalue}(A)$
    $Q := \textsc{ExtendBasis}(\{\vec{q}_1\}, \mathbb{R}^n)$ ▷ Extend $\{\vec{q}_1\}$ to a basis of $\mathbb{R}^n$ using Gram-Schmidt; see Note 13
    Unpack $Q := \begin{bmatrix} \vec{q}_1 & \widetilde{Q} \end{bmatrix}$
    Compute and unpack $Q^\top A Q = \begin{bmatrix} \lambda_1 & \vec{a}_{12}^\top \\ \vec{0}_{n-1} & \widetilde{A}_{22} \end{bmatrix}$
    $(P, \widetilde{T}) := \textsc{RealSchurDecomposition}(\widetilde{A}_{22})$
    $U := \begin{bmatrix} \vec{q}_1 & \widetilde{Q}P \end{bmatrix}$
    $T := \begin{bmatrix} \lambda_1 & \vec{a}_{12}^\top P \\ \vec{0}_{n-1} & \widetilde{T} \end{bmatrix}$
    return (U, T)
end function

Upper-Triangularization (Example)

A RLC Circuit



$$i(t) = C\frac{dV_c(t)}{dt}, \quad V_L(t) = L\frac{di(t)}{dt}$$

Stability, Controllability
Diagonalization, Triangularization

$$V_s(t) = V_R(t) + V_L(t) + V_c(t)$$

$$V_L(t) = LC\frac{d^2V_c(t)}{dt^2}, \quad V_R(t) = RC\frac{dV_c(t)}{dt}$$

$$x_1(t) = V_c(t), \quad x_2(t) = \frac{dV_c(t)}{dt}$$

$$V_s(t) = RC\frac{dV_c}{dt} + LC\underbrace{\frac{d^2V_c}{dt^2}}_{\dot{x}_2(t)} + V_c$$

$$\frac{dx_1(t)}{dt} = x_2(t)$$

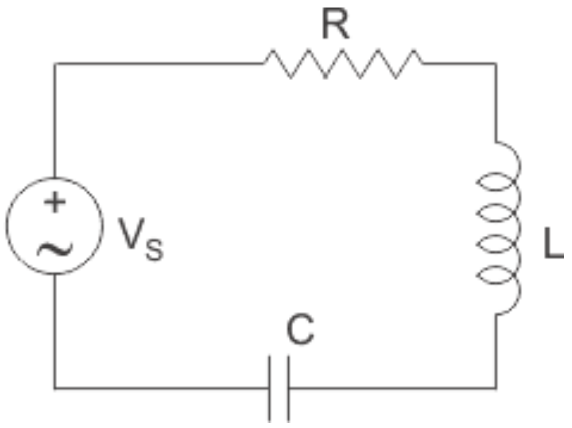
$$LC\frac{dx_2(t)}{dt} = -RC\,x_2(t) - x_1(t) + V_s(t)$$

state space

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}}^{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B} V_s$$

Upper-Triangularization (Example)

A RLC Circuit (critically damped)



$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + Bu(t)$$

$$\det(\lambda I - A) = 0 . \quad \lambda_{1,2} = \frac{-R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

Re

stability? controllable? $[AB, B]$

$$\begin{bmatrix} 1 & 0 \\ -\frac{R}{L} & 1 \end{bmatrix}$$

(1) $\lambda_1 \neq \lambda_2 < 0$ $\quad \frac{R^2}{L^2} > \frac{4}{LC}$ $\quad V = [\vec{v}_1, \vec{v}_2]$

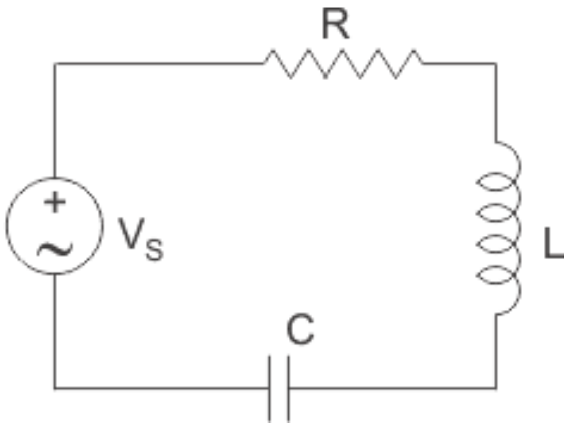
$$V^{-1} A V = \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\underbrace{I + \Delta A}_{A_d} \quad \underbrace{\Delta B}_{B_d}$$

$$y(t) = V^{-1} x(t) \quad y_1 \sim e^{-\lambda_1 t} \quad y_2 \sim e^{-\lambda_2 t}$$

Upper-Triangularization (Example)

A RLC Circuit (critically damped)



(2) $\frac{R^2}{L^2} = \frac{4}{LC}$

$\underbrace{\lambda_1 = \lambda_2}_{\lambda} = -\frac{R}{2L}$

$[\lambda I - A]\, v = 0$

$[\lambda I - A] = \begin{bmatrix} -\frac{R}{2L} & -1 \\ \frac{R^2}{4L^2} & \frac{R}{2L} \end{bmatrix}$

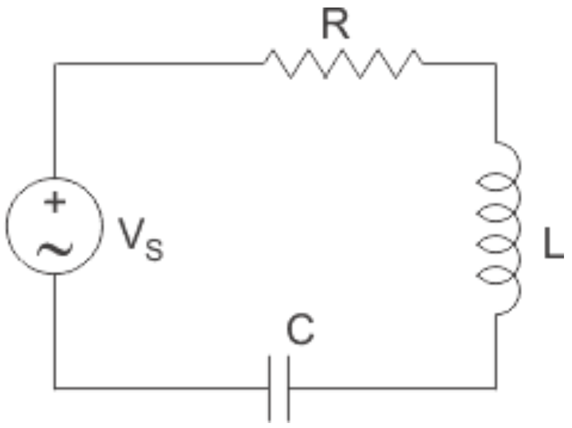
$= \begin{bmatrix} -\frac{R}{2L} & -1 \\ \frac{R}{2L}\left(\frac{R}{2L}\right) & \frac{R}{2L}(1) \end{bmatrix}$

rank $= 1$

$v = \begin{bmatrix} 1 \\ -\frac{R}{2L} \end{bmatrix}$

Upper-Triangularization (Example)

A RLC Circuit



$$U^T A U = \begin{bmatrix} \lambda & \tilde{a}_{12} \\ 0 & \lambda \end{bmatrix} \qquad \lambda = \frac{-R}{2L}$$

$$\vec{y}(t) = U^T \vec{x}(t)$$

$$\longrightarrow \begin{bmatrix} \frac{dy_1(t)}{dt} \\ \frac{dy_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} \lambda & \hat{a}_{12} \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\frac{dy_2(t)}{dt} = \lambda y_2(t) \qquad y_2(t) = y_2(0) e^{\lambda t}$$

$$U^T A U$$

$$\frac{dy_1(t)}{dt} = \lambda y_1(t) + \underline{\tilde{a}_{12} e^{\lambda t} \cdot y_2(0)}$$

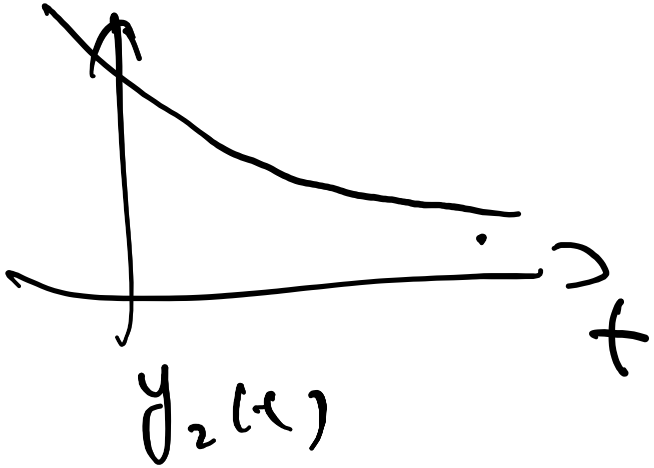
$$y_1(t) = e^{\lambda t} \cdot y_1(0) + \int_0^t \frac{e^{\lambda(t-\tau)} \tilde{a}_{12} y_2(0) e^{\lambda \tau}}{d\tau}$$

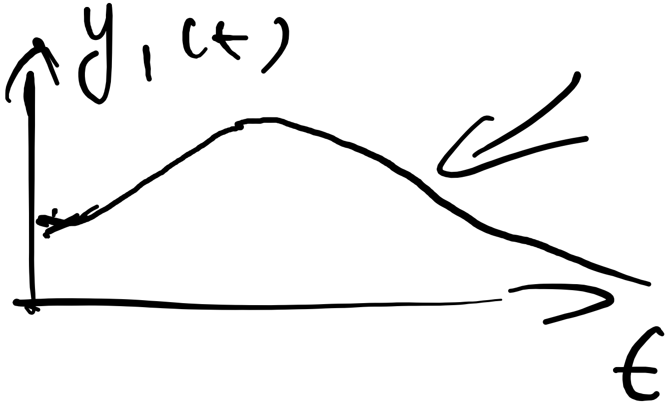
$$= \underline{e^{\lambda t} y_1(0)} + e^{\lambda t} \cdot y_2(0) \cdot t$$

$$t \cdot e^{\lambda t}$$

$e^{\lambda t}$ $\quad t e^{\lambda t}$

$\lambda < 0$



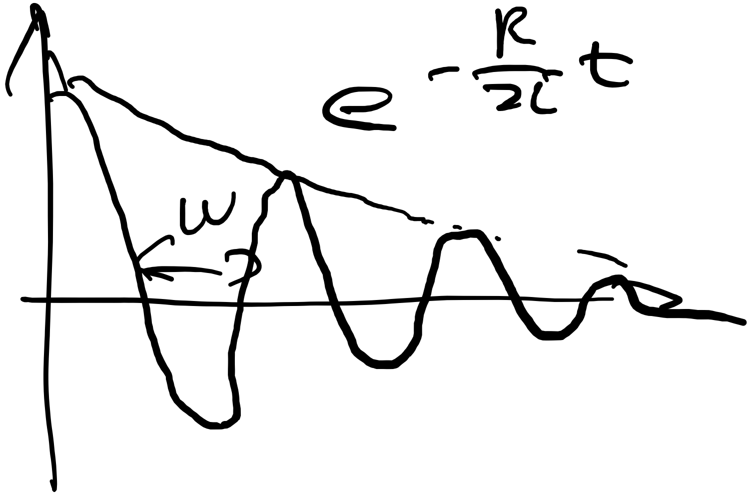


③ $\frac{R^2}{L^2} \;\substack{>\\=\\<}\; \frac{4}{LC}$

$t^n e^{\lambda t}$

$\lambda_{1,2} = -\frac{R}{2L} \pm j\omega$

$e^{-\frac{R}{2L}t}$



Spectral Theorem (motivations)

Diagonalization for $A \in \mathbb{R}^{n\times n}$ with n independent eigenvectors: $V^{-1}AV = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$

$[v_1, \cdots, v_n]$

Triangularization for $A \in \mathbb{R}^{n\times n}$ with real eigenvalues: $U^{-1}AU = U^\top AU = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$

$U^T U = I$

For real symmetric matrices $A = A^\top \in \mathbb{R}^{n\times n}$:

$$M = \underbrace{\frac{M + M^T}{2}}_{\text{sym}} + \underbrace{\frac{M - M^T}{2}}_{\text{anti-sym}}$$

$$V^{-1}AV = V^\top AV = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

Spectral Theorem (statement)

Theorem: Let $A = A^\top \in \mathbb{R}^{n\times n}$ be a *real and symmetric* matrix. Then

1. All eigenvalues of A are real.
2. A is diagonalizable.
3. All eigenvectors are orthogonal to each other.

proof: 1. $(\lambda, \vec{v})$ $\quad A\vec{v} = \lambda \vec{v}$ ←

$$\bar{\vec{v}}^T \bar{A}^T = \bar{\lambda}\, \bar{\vec{v}}^T \qquad \bar{\vec{v}}^T A = \bar{\lambda}\, \bar{\vec{v}}^T$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\bar{\vec{v}}^T A \vec{v} = \bar{\lambda}\, \bar{\vec{v}}^T \vec{v}$$

$$\bar{\vec{v}}^T \vec{v} = \bar{v}_1 v_1 + \bar{v}_2 v_2 + \cdots + \bar{v}_n v_n = |v_1|^2 + |v_2|^2 + \cdots + |v_n|^2$$

Real.

$$\bar{\vec{v}}^T \lambda \vec{v} = \bar{\lambda}\, \bar{\vec{v}}^T \vec{v}$$

$$\lambda\, \bar{\vec{v}}^T \vec{v} = \bar{\lambda}\, \bar{\vec{v}}^T \vec{v} \Rightarrow \lambda = \bar{\lambda} \quad \lambda\text{ - real} \quad \blacksquare$$

Spectral Theorem (proof)

② $U^T U = I \quad \underline{U^T A U = T}$

$$T^T = (U^T A U)^T = U^T A^T \underline{(U^T)^T} = \underline{U^T A U} = T$$

$T^T = T \qquad \underline{T\text{ - diagonal}}$ $\boxed{\S}$

③ $U^T A U = \Lambda = T$

$AU = U\Lambda \qquad U^T U = I$ $\boxed{\S}$

Spectral Theorem (extensions)

Consider: $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) \quad \text{with } A \text{ symmetric, and } \lambda_{\max}(A) < -\lambda.$

How does the "energy" $V(t) = \|\vec{x}(t)\|_2^2 = \vec{x}(t)^\top \vec{x}(t)$ evolve?

Spectral Theorem (extensions)

What if A is real and *anti-symmetric*: $A^\top = -A \in \mathbb{R}^{n\times n}$