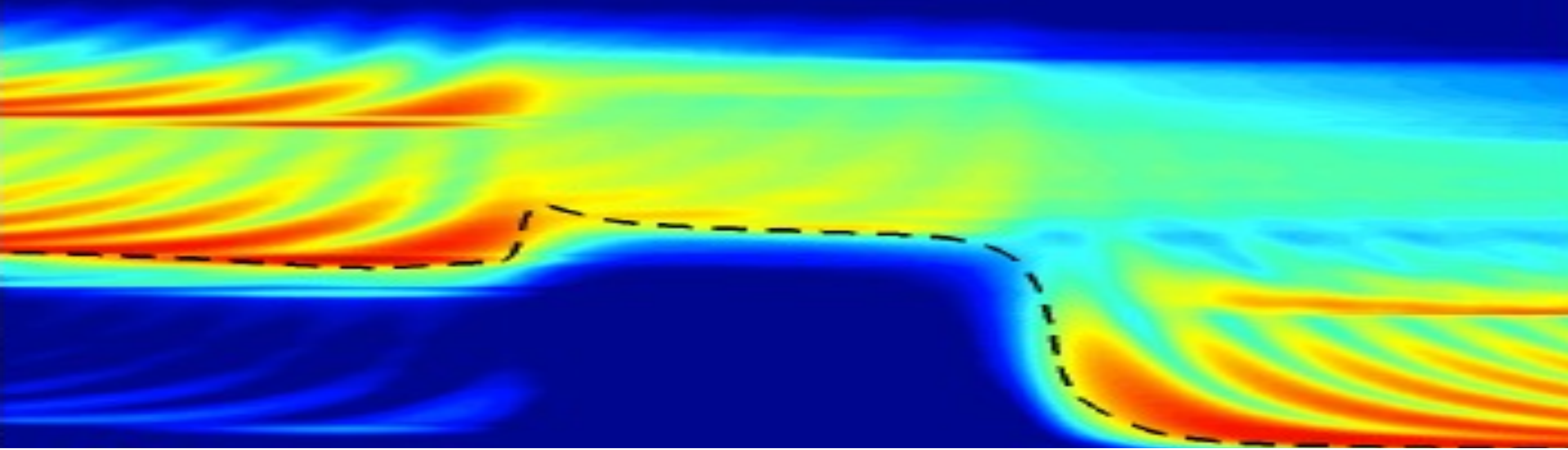

# EECS 16B
# Designing Information Devices and Systems II
# Lecture 22

Prof. Yi Ma

Department of Electrical Engineering and Computer Sciences, UC Berkeley,
yima@eecs.berkeley.edu

# Outline

- Singular Value Decomposition (SVD)
  - Theorem (with proof)
  - Examples of SVD
  - Full SVD
  - Geometric Interpretation of SVD

# Singular Value Decomposition (SVD)

Given $A \in \mathbb{R}^{m \times n}$ with $\mathrm{rank}(A) = r$ , we like to decompose it into a special matrix form:

$$U_r = [\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r] \text{ orthogonal}$$

$$V_r = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r] \text{ orthogonal}$$

$$\Sigma_r = \mathrm{diag}\{\sigma_1, \sigma_2, \dots, \sigma_r\} > 0$$

$$A = U_r \Sigma_r V_r^\top = [\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vdots \\ \vec{v}_r^\top \end{bmatrix}$$

# Singular Value Decomposition (Theorem)

**Theorem:** given $A \in \mathbb{R}^{m \times n}$ with $\mathrm{rank}(A) = r$ , let $A^\top A = \sum_{i=1}^{r} \lambda_i \vec{v}_i \vec{v}_i^\top$ and $\sigma_i = \sqrt{\lambda_i}$ ,

$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i \in \mathbb{R}^m, \ i = 1, \ldots, r.$ Then we have $U_r = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r]$ $\mathrm{orthogonal}$ , and

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top = U_r \Sigma_r V_r^\top \qquad \Sigma_r = \mathrm{diag}\{\sigma_1, \ldots, \sigma_r\} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_r \end{bmatrix}$$

**Proof:**

# Singular Value Decomposition (Theorem)

**Theorem:** given $A \in \mathbb{R}^{m \times n}$ with $\mathrm{rank}(A) = r$, let $A^\top A = \sum_{i=1}^{r} \lambda_i \vec{v}_i \vec{v}_i^\top$ and $\sigma_i = \sqrt{\lambda_i}$, $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i \in \mathbb{R}^m, \; i = 1, \ldots, r$. Then we have $U = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r]$ orthogonal, and

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top = U_r \Sigma_r V_r^\top$$

**Proof:**

# Singular Value Decomposition

Given $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = r$, two (equivalent) ways to find SVD:

$A^\top A \in \mathbb{R}^{n \times n}$

$AA^\top \in \mathbb{R}^{m \times m}$

# Singular Value Decomposition (example)

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}, \ A^\top = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}$$

# Singular Value Decomposition (example)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; A^\top = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

# Compact versus Full SVD

Compact SVD: $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top = U_r \Sigma_r V_r^\top$ $\quad A = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vdots \\ \vec{v}_r^\top \end{bmatrix}$

# Compact versus Full SVD

Full SVD: $A = U\Sigma V^\top = [\vec{u}_1, \ldots, \vec{u}_r, \vec{u}_{r+1}, \ldots, \vec{u}_m] \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \sigma_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vdots \\ \vec{v}_r^\top \\ \vec{v}_{r+1}^\top \\ \vdots \\ \vec{v}_n^\top \end{bmatrix}$

# Full SVD for Full-rank Matrices

# Geometric Interpretation of SVD

# Geometric Interpretation of SVD

$$A = U \quad \Sigma \quad V^\top$$