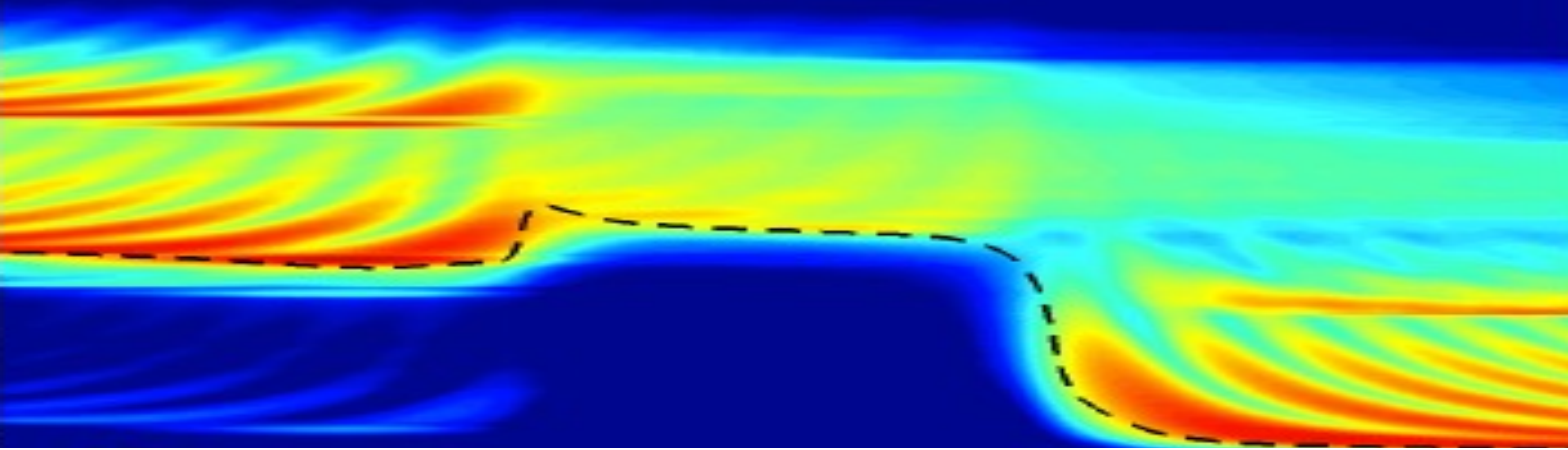

EECS 16B
Designing Information Devices and Systems II
Lecture 23

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Outline

- Singular Value Decomposition (SVD)
  - Geometric Interpretation of SVD
- Applications of SVD: unifying
  - Matrix (Pseudo) Inverse
  - Least Squares
  - Minimum Norm Solution

Singular Value Decomposition (SVD)

Given $A \in \mathbb{R}^{m \times n}$ with $\mathrm{rank}(A) = r$, we like to decompose it into a special matrix form:

$V = [\vec{v}_1, \ldots, \vec{v}_n]$ orthonormal e.v.'s for $A^\top A$

$U = [\vec{u}_1, \ldots, \vec{u}_m]$ orthonormal e.v.'s for $AA^\top$

eigenvalues of $A^\top A$ (or $AA^\top$) : $\lambda_1 \geq \cdots \geq \lambda_r > 0 \cdots 0$

$\Sigma_r = \mathrm{diag}\{\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}\} > 0$

Compact SVD:

$$A = U_r \Sigma_r V_r^\top = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vdots \\ \vec{v}_r^\top \end{bmatrix}$$

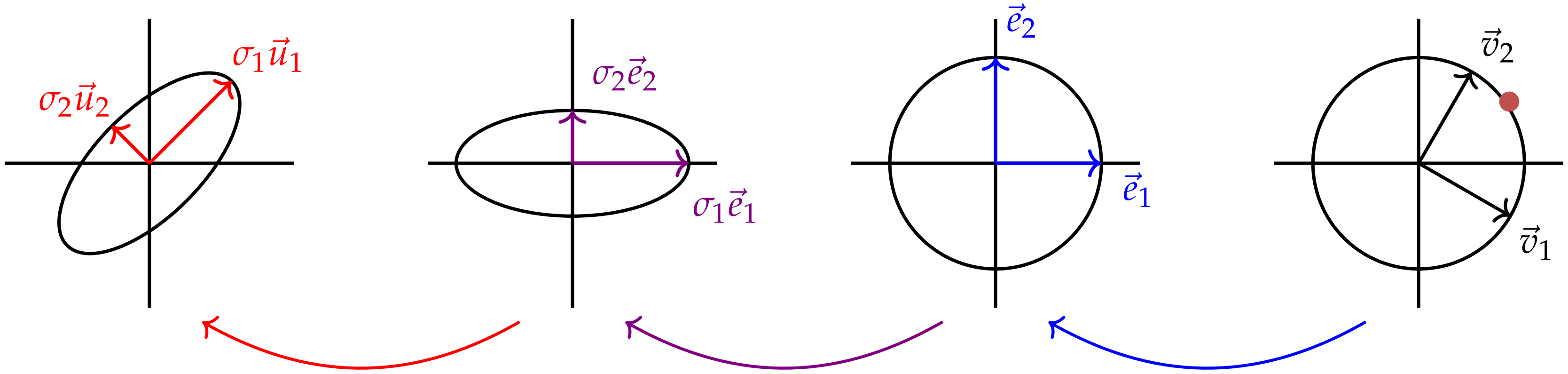
Full SVD:

$$A = U \Sigma V^\top = [\vec{u}_1, \ldots, \vec{u}_r, \vec{u}_{r+1}, \ldots, \vec{u}_n] \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \sigma_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vdots \\ \vec{v}_r^\top \\ \vec{v}_{r+1}^\top \\ \vdots \\ \vec{v}_n^\top \end{bmatrix}$$

Geometric Interpretation of SVD

$$\vec{y} = A\vec{x}: \;\; A = U\Sigma V^\top = [U_r, U_{m-r}] \begin{bmatrix} \Sigma_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} \begin{bmatrix} V_r^\top \\ V_{n-r}^\top \end{bmatrix}$$

$$A = \quad U \quad \Sigma \quad V^\top$$



“rotation” scaling “rotation”

Geometric Interpretation of SVD (Example)

$$A = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}$$

Algebraic Interpretation of SVD

$$A = U\Sigma V^\top = [U_r, U_{m-r}] \begin{bmatrix} \Sigma_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} \begin{bmatrix} V_r^\top \\ V_{n-r}^\top \end{bmatrix} = U_r \Sigma_r V_r^\top$$

Applications of SVD: Matrix Inverse

Given $A \in \mathbb{R}^{m \times n}$ with $\mathrm{rank}(A) = r = m = n :\ A = U\Sigma V^\top$. What is its inverse?

Applications of SVD: Matrix Pseudo Inverse

Definition: Given $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = r$ and SVD:

$$A = U\Sigma V^\top = U \begin{bmatrix} \Sigma_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} V^\top$$

its (Moore-Penrose) **pseudo inverse** is defined to be:

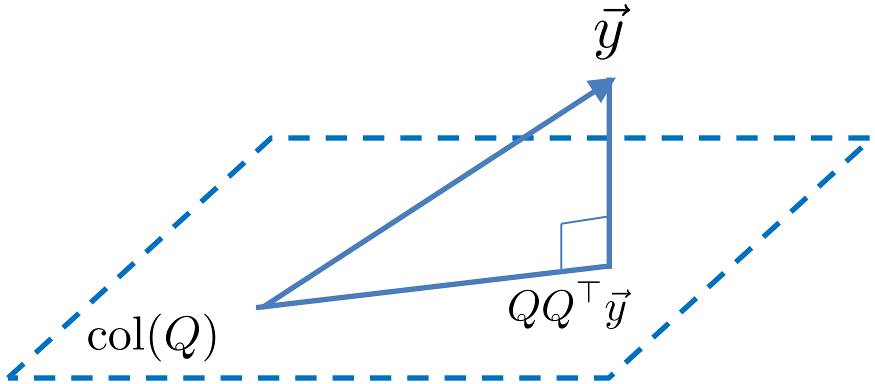
$$A^\dagger = V \begin{bmatrix} \Sigma_r^{-1} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} U^T = V_r \Sigma_r^{-1} U_r^\top$$

Example: $A = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}, \quad A^\dagger = ?$

Applications of SVD: Matrix Pseudo Inverse

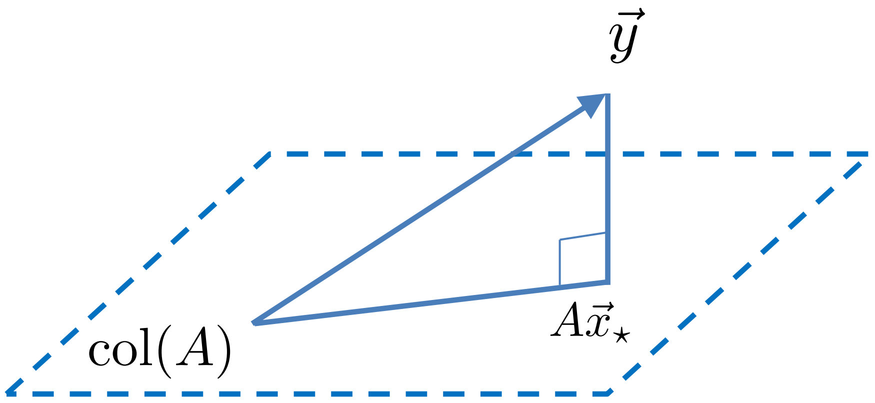
Applications of SVD: Matrix Pseudo Inverse

Geometric interpretation of $AA^\dagger$ or $A^\dagger A$.



Applications of SVD: Least Squares

$$\min_{\vec{x}} \|\vec{y} - A\vec{x}\|_2^2, \text{ with } A \in \mathbb{R}^{m\times n} \text{ and } \mathrm{rank}(A) = n : \quad \vec{x}_\star = (A^\top A)^{-1} A^\top \vec{y}$$



Applications of SVD: Least Squares

Show: Given $A \in \mathbb{R}^{m \times n}$ and $\text{rank}(A) = n$: $A^{\dagger} = (A^{\top}A)^{-1}A^{\top}$

Applications of SVD: Minimum Norm Solution

$$\min_{\vec{x}} \|\vec{x}\|_2^2 \text{ s.t. } \vec{y} = A\vec{x}, \text{ with } A \in \mathbb{R}^{m \times n} \text{ and } \text{rank}(A) = m : \quad \vec{x}_\star = A^\top (AA^\top)^{-1} \vec{y}$$

Show: $\vec{x}_\star = A^\dagger \vec{y} \; (= A^\top (AA^\top)^{-1} \vec{y})$.

Applications of SVD: Minimum Norm Solution