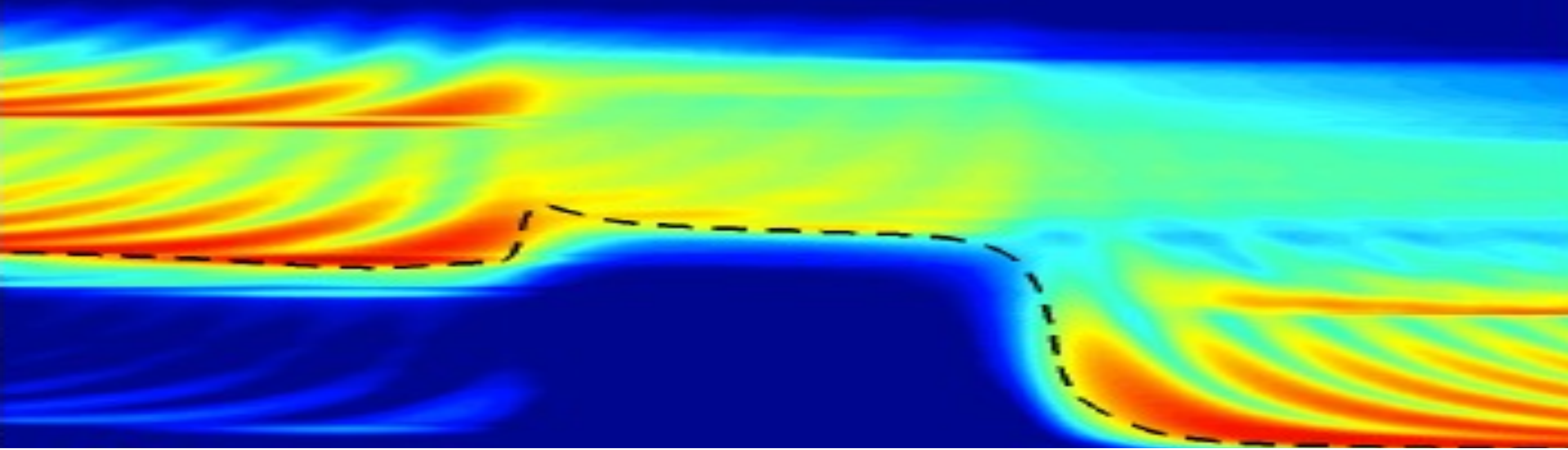

EECS 16B
Designing Information Devices and Systems II
Lecture 26

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Outline

- Linearization of Nonlinear Control Systems
  - Operating points
  - Scalar case and an example
  - Vector case and an example

Nonlinear Autonomous Systems: Equilibrium Points

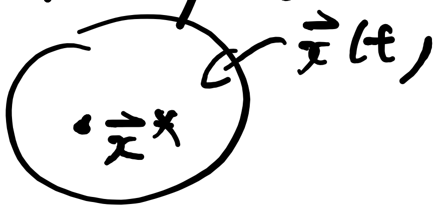
$$\frac{d\vec{x}(t)}{dt} = \vec{f}(\vec{x}(t)) \in \mathbb{R}^n$$

$$\vec{x}[i+1] = \vec{f}(\vec{x}[i]) \in \mathbb{R}^n$$

$\vec{x}(t)$ constant.

$\vec{f}(\vec{x}^*) = 0$ $\vec{x}^*$ equilibrium pt.

$\vec{x}(t)$ around $\vec{x}^*$



$$\vec{x}^* = \vec{f}(\vec{x}^*)$$

$$\vec{\delta}[i] = \vec{x}(t) - \vec{x}^*$$

$$\vec{\delta}[i+1] = \frac{\partial \vec{f}}{\partial \vec{x}}(\vec{x}^*)\,\vec{\delta}[i]$$

$\vec{\delta}x(t) = \vec{x}(t) - \vec{x}^*$ small

$$\frac{d\vec{\delta}x(t)}{dt} = \vec{f}(\vec{x}(t)) = \underbrace{\vec{f}(x^*)}_{=0} + \overbrace{\boxed{\frac{\partial \vec{f}}{\partial \vec{x}}(\vec{x}^*)}}^{A}\overbrace{(\vec{x}(t) - \vec{x}^*)}^{\vec{\delta}x(t)} + \text{h.o.t.}$$

$$\frac{d\vec{\delta}x(t)}{dt} \approx \boxed{A}\,\vec{\delta}x(t) \leftarrow \text{stable?}$$

Nonlinear Control Systems: Operating Points

$$\frac{d\vec{x}(t)}{dt} = \vec{f}(\vec{x}(t), \vec{u}(t)) \in \mathbb{R}^n \Leftarrow$$

$(\vec{x}^*, \vec{u}^*)$ operating point
if $\vec{f}(\vec{x}^*, \vec{u}^*) = 0$

$$\vec{x}[i+1] = \vec{f}(\vec{x}[i], \vec{u}[i]) \in \mathbb{R}^n$$

$(\vec{x}^*, \vec{u}^*)$ is an operating pt.
if $\vec{x}^* = \vec{f}(\vec{x}^*, \vec{u}^*)$

* $(\vec{x}^*, \vec{u}^*)$ may not be unique. $\vec{f}(\vec{x}^*, \underline{\vec{u}^*}) = 0 \in \mathbb{R}^n \Rightarrow \vec{u}^* = g(\vec{x}^*)$

$\in \mathbb{R}^m$ $\quad m \leq n$

* linearized system around $(\vec{x}^*, \vec{u}^*)$.

Nonlinear Control Systems: Linearization

Scalar case: $\frac{dx(t)}{dt} = f(x(t), u(t)) \;\in \mathbb{R}$

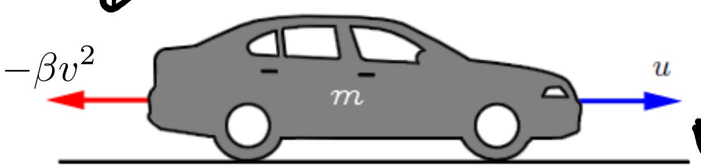
operating point $f(x^*, u^*) = 0$ $\qquad \delta x(t) = x(t) - x^*$

$$\frac{d\,\delta x(t)}{dt} = f(x(t), u(t)) = \underbrace{f(x^*, u^*)}_{0} + \overbrace{\frac{\partial f}{\partial x}(x^*, u^*)}^{\lambda}\overbrace{(x(t) - x^*)}^{\delta x(t)} + \underbrace{\frac{\partial f}{\partial u}(x^*, u^*)}_{b}\underbrace{(u(t) - u^*)}_{\delta u(t)} + h.o.t.$$

$$\frac{d\,\delta(t)}{dt} = \lambda\,\delta x(t) + b\,\delta u(t)$$

Nonlinear Control Systems: Example

d



$$\longrightarrow \frac{dx(t)}{dt} = -\frac{\beta}{m}x(t)^2 + \frac{1}{m}u(t) = \underline{f(x(t), u(t))}$$

$$m\frac{v(t)}{dt} = -\beta v(t)^2 + u(t)$$

$$ma = F$$

$x(t) = v(t)$ $\quad x^* = u^*$ $\quad f(x^*, u^*) = 0$

$$-\frac{\beta}{m}x^{*2} + \frac{1}{m}u^* = 0 \Rightarrow \underline{u^* = \beta x^{*2}}$$

linearize at (x^*, u^*)

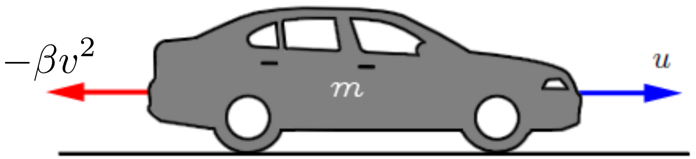
$$\left.\frac{\partial f}{\partial x}\right|_{x^*,u^*} = \underbrace{-\frac{2\beta}{m}x^*}_{\lambda}, \quad \left.\frac{\partial f}{\partial u}\right|_{x^*,u^*} = \underbrace{\frac{1}{m}}_{b}$$

$$\frac{d\,\delta x(t)}{dt} = \lambda\,\delta x(t) + \underline{b\,\delta u(t)}$$

$$\delta u(t) = 0 \Rightarrow u(t) \equiv u^*$$

$$\delta x(t) = e^{-\frac{2\beta}{m}x^* t}\,\delta x(0) \longrightarrow \frac{d\,\delta x(t)}{dt} = \underline{\lambda}\,\delta x(t) \text{ — } \underline{\text{stable}}$$

Nonlinear Control Systems: Example



$$\frac{dx(t)}{dt} = -\frac{\beta}{m}x(t)^2 + \frac{1}{m}u(t) = f(x(t), u(t))$$

$$\frac{d\,\delta x(t)}{dt} = \lambda\,\delta x(t) + b\,\delta u(t) \Leftarrow \boxed{\delta u(t) = k\,\delta x(t)}$$

$$= \underbrace{(\lambda + bk)}_{\lambda'}\delta x(t) \qquad \lambda' = \lambda + bk = -\frac{2\beta x^*}{m} + \frac{k}{m}$$

$k < 0$

$$\delta x(t) = e^{\lambda' t}\,\delta x(0) \qquad \lambda'$$

$$u(t) = u^* + k\,\delta x(t) = u^* + k(x(t) - x^*)$$

$$= \boxed{\underbrace{\beta x^{*2}}_{u^*}} + \boxed{k(x(t) - x^*)} \Leftarrow \text{cruise control}$$

Nonlinear Control Systems: Linearization

Vector case: $\underline{\frac{d\vec{x}(t)}{dt} = \vec{f}(\vec{x}(t), \vec{u}(t)) \in \mathbb{R}^n}$ with $\underline{\vec{x}(t) \in \mathbb{R}^n}$, $\underline{\vec{u}(t) \in \mathbb{R}^m}$

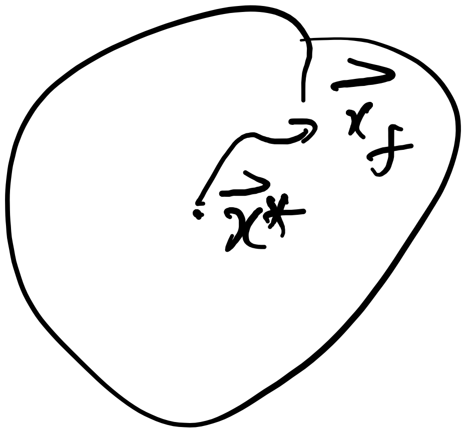
$$\vec{f}(\vec{x}^*, \vec{u}^*) = 0 \qquad \vec{\delta}x(t) = \vec{x}(t) - \vec{x}^*, \qquad \vec{\delta}u(t) = \vec{u}(t) - \vec{u}^*$$

$$\frac{d\vec{\delta}x(t)}{dt} \approx \underbrace{\frac{\partial \vec{f}}{\partial \vec{x}}(\vec{x}^*, \vec{u}^*)}_{A} \vec{\delta}x(t) + \frac{\partial \vec{f}}{\partial \vec{u}}(\vec{x}^*, u^*) \delta_u(t) + h.o.t.$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}^*, \vec{u}^*) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{x}^*, \vec{u}^*) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{x}^*, \vec{u}^*) & \cdots & \frac{\partial f_n}{\partial x_n}(\vec{x}^*, \vec{u}^*) \end{bmatrix}_{n \times n}, \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{n \times m}$$

$$\frac{\vec{\delta x}(t)}{dt} = A\,\vec{\delta x}(t) + B\,\vec{\delta u}(t)$$

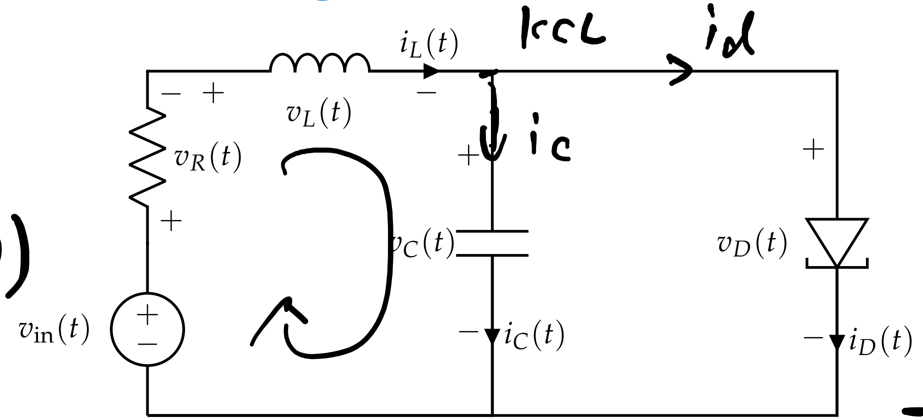
linear time-invariant



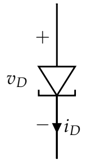
$$\vec{\delta x}_f = \vec{x}_f - \vec{x}_*$$

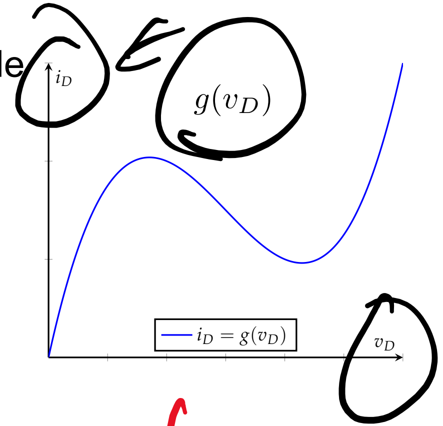
Nonlinear Control Systems: Linearization

Example: $\dfrac{d\vec{x}(t)}{dt} = \vec{f}(\vec{x}(t), \vec{u}(t))$



a tunnel diode





$$\begin{cases} \dfrac{dV_c(t)}{dt} = \dfrac{1}{C}\left(i_L(t) - g(V_c(t))\right) \\[2ex] \dfrac{di_L(t)}{dt} = \dfrac{1}{L}\left(V_{in} - R\,i_L(t) - V_c(t)\right) \end{cases}$$

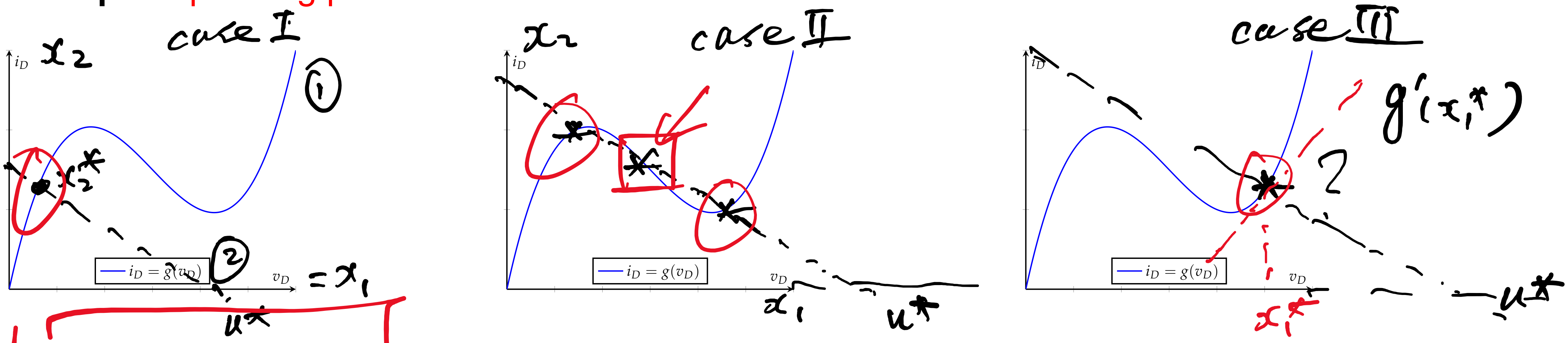
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} V_c \\ i_L \end{bmatrix}, \quad u = V_{in}$$

$$\begin{cases} \dfrac{dx_1(t)}{dt} = \overbrace{\dfrac{1}{C}\left(x_2 - g(x_1)\right)}^{f_1} \quad (=0) \\[2ex] \dfrac{dx_2(t)}{dt} = \underbrace{\dfrac{1}{L}\left(u - R x_2 - x_1\right)}_{\vec{f}(\vec{x},u)}^{f_2} \quad (=0) \end{cases}$$

$$\vec{f}(\vec{x}^*, u^*) = 0 \in \mathbb{R}^2$$

Nonlinear Control Systems: Linearization

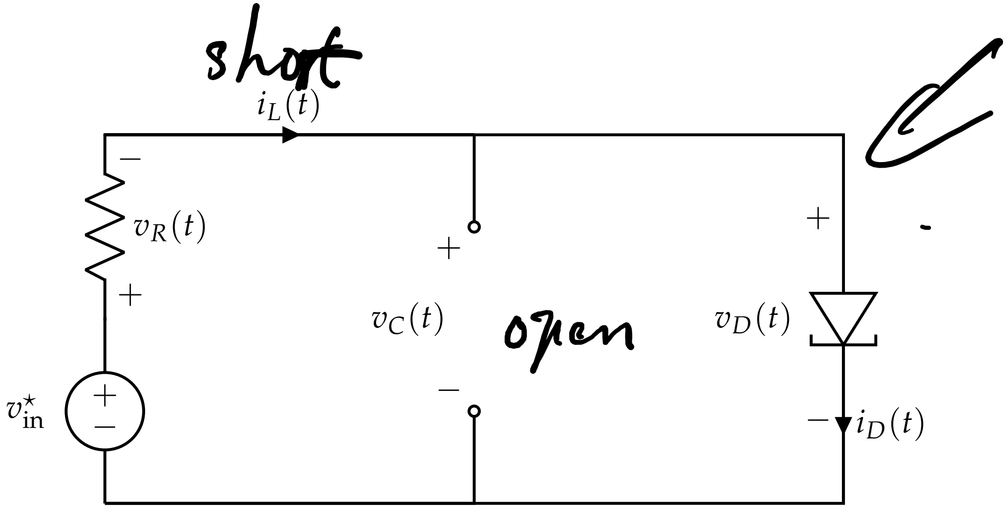
Example: operating points.



$$\boxed{\begin{aligned} f_1(\vec{x}^*, u^*) &= 0 \\ f_2(\vec{x}^*, u^*) &= 0 \end{aligned}} \quad \begin{aligned} &\Rightarrow x_2^* = g(x_1^*) \quad (1) \\ &\Rightarrow \underline{x_2^* = \frac{u^* - x_1^*}{R}} \quad (2) \end{aligned}$$

Nonlinear Control Systems: Linearization

Example: interpretation of operating points.



$$\vec{f}(\bar{x}, u) = 0$$

$$\left.\frac{d\vec{x}(t)}{dt}\right|_{x^*, u^*} = f(\vec{x}^*, u^*) = 0$$

$$\frac{d\,v_c(t)}{dt} = 0 \Rightarrow i_c(t) = 0$$

$$\frac{d\,i_L(t)}{dt} = 0 \Rightarrow v_L(t) = 0$$

equilibrium

"steady state"

Nonlinear Control Systems: Linearization

Example: linearized system.

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -\frac{g'(x_1^*)}{C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \in \mathbb{R}^{1}$$

$$\frac{d\vec{\delta}x(t)}{dt} = A\,\vec{\delta}x(t) + B\,\delta u(t) \Longleftarrow \text{LTI } \underline{\text{system}}$$

Nonlinear Control Systems: Linearization

Example: stability and controllability of the linearized system.

$$A = \begin{bmatrix} -\frac{g'(x_1^*)}{C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}$$

$$(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - \underbrace{(\lambda_1+\lambda_2)}_{tr(A)}\lambda + \underbrace{\lambda_1\cdot\lambda_2}_{det(A)}$$

$$tr(A) < 0 \qquad det(A) > 0$$

$$\left.\begin{aligned} tr(A) &= -\frac{g'(x_1^*)}{C} - \frac{R}{L} < 0 \\ det(A) &= \frac{g'(x_1^*)R}{LC} + \frac{1}{LC} > 0 \end{aligned}\right\} \text{ if } g'(x_1^*) > 0$$

$$g'(x_1^*) < 0$$

$$A = \begin{bmatrix} \frac{-g'(x_1^*)}{C} & \frac{1}{C} \\ -\frac{1}{L} & \frac{-R}{L} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

$$\frac{d\vec{\delta x}(t)}{dt} = A\vec{\delta x}(t) + B\vec{\delta u}(t)$$

controllable ?

$$C = [AB, B] \quad \text{rank } 2?$$

$$= \begin{bmatrix} \frac{1}{CL} & 0 \\ -\frac{R}{L^2} & \frac{1}{L} \end{bmatrix}$$

— yes!

$$\delta u(t) = k\vec{\delta x}(t)$$