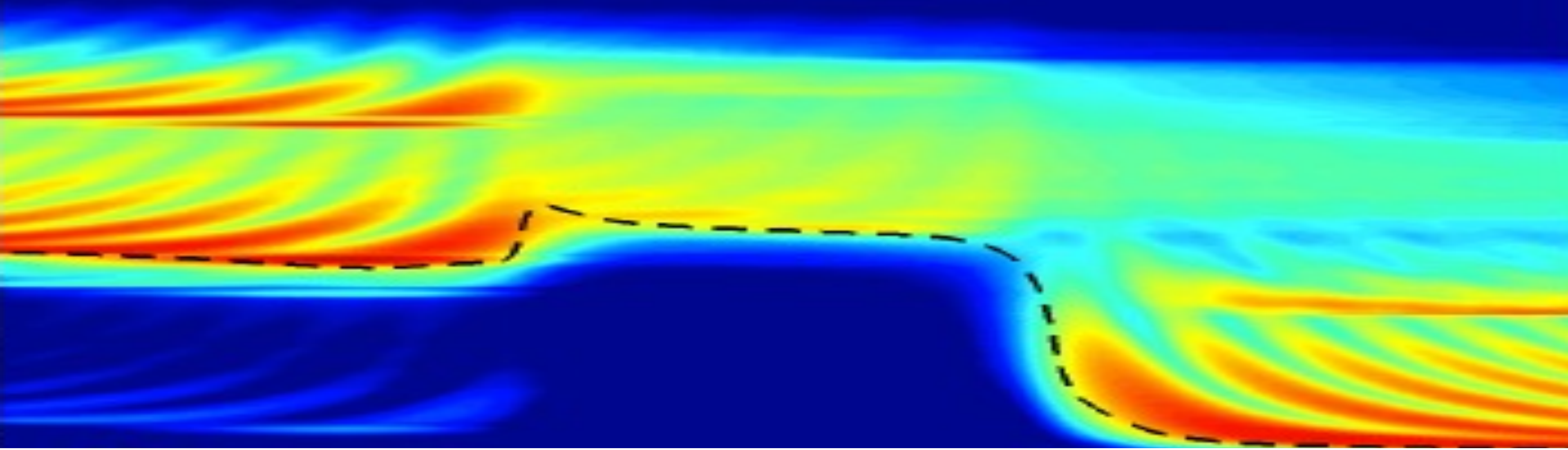

EECS 16B
Designing Information Devices and Systems II
Lecture 27

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Outline

- Complex Linear Algebra
  - Complex linear vector space
  - Norm and inner product
  - Unitary matrix, Hermitian matrix
  - Gram-Schmidt, Schur Decomposition, SVD
  - Least Squares and Minimum Norm Solutions

Vector Space

A **vector space:** $(\mathbb{V}, \mathbb{F})$ is closed under vector addition and scalar multiplication:

$$\forall \vec{v}_1, \vec{v}_2 \in \mathbb{V}, \text{ and } \forall \alpha, \beta \in \mathbb{F} \qquad \alpha \cdot \vec{v}_1 + \beta \cdot \vec{v}_2 \in \mathbb{V}$$

The addition is associative and commutative; there is an identity/zero vector $\vec{0}$, and every vector has an inverse.

The multiplication is associative, commutative, and distributive; there is an identity scalar **1**.

A **norm** $\|\cdot\|$ on the vector space satisfies:

$$\|\vec{x}\| \geq 0 \quad \forall \vec{x} \in \mathbb{V} \quad \text{and} \quad \|\vec{x}\| = 0 \Leftrightarrow \vec{x} = \vec{0}$$

$$\|\alpha\vec{x}\| = |\alpha| \cdot \|\vec{x}\| \quad \forall \vec{x} \in \mathbb{V}, \; \alpha \in \mathbb{F}$$

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|, \quad \forall \vec{x}, \vec{y} \in \mathbb{V}$$

Real versus Complex Vector Space

$$(\mathbb{V},\mathbb{F}) = (\mathbb{R}^n,\mathbb{R})$$

real vector transpose

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad \vec{x}^\top \doteq [x_1, x_2, \ldots, x_n]$$

Inner product: $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i = \vec{y}^\top \vec{x} = \vec{x}^\top \vec{y}$

2-norm: $\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle = \vec{x}^\top \vec{x} = \sum_{i=1}^{n} x_i^2$

$$\|\vec{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$(\mathbb{V},\mathbb{F}) = (\mathbb{C}^n,\mathbb{C})$$

(note2j)

complex conjugate transpose

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n, \quad \vec{x}^* \doteq \left(\bar{\vec{x}}\right)^\top = [\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n]$$

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i \bar{y}_i = \vec{y}^* \vec{x} \quad \left(= \overline{\vec{x}^* \vec{y}} = \overline{\langle \vec{y}, \vec{x} \rangle} \right)$$

$$\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle = \vec{x}^* \vec{x} = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2$$

$$\|\vec{x}\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

Complex Vector Norm and Inner Product

Real versus Complex Matrix

$$(\mathbb{V}, \mathbb{F}) = (\mathbb{R}^n, \mathbb{R})$$

real matrix transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m,n-1} & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$A^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{1n} & \cdots & a_{m-1,n} & a_{mn} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$(\mathbb{V}, \mathbb{F}) = (\mathbb{C}^n, \mathbb{C})$$

complex conjugate transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m,n-1} & a_{mn} \end{bmatrix} \in \mathbb{C}^{m \times n}$$

$$A^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{m1} \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ \bar{a}_{1n} & \cdots & \bar{a}_{m-1,n} & \bar{a}_{mn} \end{bmatrix} \in \mathbb{C}^{n \times m}$$

Complex Matrices

Algebraic manipulations, row, column, null space, rank, inverse, eigenvectors and eigenvalues are all similar to those of real matrices.

Real versus Complex Matrices

$$(\mathbb{V}, \mathbb{F}) = (\mathbb{R}^n, \mathbb{R})$$

Orthogonal Matrix: $Q = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n] \in \mathbb{R}^{n \times n}$

$$\vec{q}_i^\top \vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \quad \text{(orthogonal)} \\ 1 & \text{if } i = j \quad \text{(normalized)} \end{cases}$$

$$Q^\top Q = I = QQ^\top$$

$$(\mathbb{V}, \mathbb{F}) = (\mathbb{C}^n, \mathbb{C})$$

Unitary Matrix: $Q = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n] \in \mathbb{C}^{n \times n}$

$$\vec{q}_i^* \vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \quad \text{(orthogonal)} \\ 1 & \text{if } i = j \quad \text{(normalized)} \end{cases}$$

$$Q^* Q = I = QQ^*$$

Gram-Schimdt Orthonormalization (QR)

$$D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] \in \mathbb{R}^{n \times k}$$ **(Lecture 17)**

QR: $$[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$$

Gram-Schimdt:

$$\vec{z}_1 = \vec{d}_1 \qquad \vec{q}_1 = \vec{z}_1 / \|\vec{z}_1\|$$

$$\vec{z}_2 = \vec{d}_2 - (\vec{d}_2^\top \vec{q}_1)\vec{q}_1 \qquad \vec{q}_2 = \vec{z}_2 / \|\vec{z}_2\|$$

$$\vec{z}_3 = \vec{d}_3 - (\vec{d}_3^\top \vec{q}_1)\vec{q}_1 - (\vec{d}_3^\top \vec{q}_2)\vec{q}_2 \qquad \vec{q}_3 = \vec{z}_3 / \|\vec{z}_3\|$$

$$\vdots \qquad \vdots$$

$$\vec{z}_k = \vec{d}_k - \sum_{j=1}^{k-1} (\vec{d}_k^\top \vec{q}_j)\vec{q}_j \qquad \vec{q}_k = \vec{z}_k / \|\vec{z}_k\|$$

Gram-Schimdt Orthonormalization (QR)

$$D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] \ \in \mathbb{C}^{n \times k}$$

QR:

$$[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$$

Gram-Schimdt:

$$\vec{z}_1 = \vec{d}_1 \qquad \vec{q}_1 = \vec{z}_1 / \|\vec{z}_1\|$$

$$\vec{z}_2 = \vec{d}_2 - \langle \vec{d}_2, \vec{q}_1 \rangle \vec{q}_1 \qquad \vec{q}_2 = \vec{z}_2 / \|\vec{z}_2\|$$

$$\vec{z}_3 = \vec{d}_3 - \langle \vec{d}_3, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{d}_3, \vec{q}_2 \rangle \vec{q}_2 \qquad \vec{q}_3 = \vec{z}_3 / \|\vec{z}_3\|$$

$$\vdots \qquad \vdots$$

$$\vec{z}_k = \vec{d}_k - \sum_{j=1}^{k-1} \langle \vec{d}_k, \vec{q}_j \rangle \vec{q}_j \qquad \vec{q}_k = \vec{z}_k / \|\vec{z}_k\|$$

Schur Decomposition (Upper Triangularization)

$A \in \mathbb{R}^{n\times n}$ **(Lecture 18)**

$$T = U^{-1}AU = U^\top AU = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

Algorithm 10 Real Schur Decomposition

Input: A square matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues.

Output: An orthonormal matrix $U \in \mathbb{R}^{n\times n}$ and an upper-triangular matrix $T \in \mathbb{R}^{n\times n}$ such that $A = UTU^\top$.

function $\textsc{RealSchurDecomposition}(A)$
    if A is 1×1 **then**
        return $\begin{bmatrix}1\end{bmatrix}, A$
    end if
    $(\vec{q}_1, \lambda_1) := \textsc{FindEigenvectorEigenvalue}(A)$
    $Q := \textsc{ExtendBasis}(\{\vec{q}_1\}, \mathbb{R}^n)$ ▷ Extend $\{\vec{q}_1\}$ to a basis of $\mathbb{R}^n$ using Gram-Schmidt; see Note 13
    Unpack $Q := \begin{bmatrix} \vec{q}_1 & \widetilde{Q} \end{bmatrix}$
    Compute and unpack $Q^\top AQ = \begin{bmatrix} \lambda_1 & \vec{a}_{12}^\top \\ \vec{0}_{n-1} & \widetilde{A}_{22} \end{bmatrix}$
    $(P, \widetilde{T}) := \textsc{RealSchurDecomposition}(\widetilde{A}_{22})$
    $U := \begin{bmatrix} \vec{q}_1 & \widetilde{Q}P \end{bmatrix}$
    $T := \begin{bmatrix} \lambda_1 & \vec{a}_{12}^\top P \\ \vec{0}_{n-1} & \widetilde{T} \end{bmatrix}$
    return (U, T)
end function

Schur Decomposition (Upper Triangularization)

$$A \in \mathbb{C}^{n\times n}$$

$$T = U^{-1}AU = U^*AU = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

Algorithm 64 Schur Decomposition

Input: A square matrix $A \in \mathbb{C}^{n\times n}$.

Output: A unitary matrix $U \in \mathbb{C}^{n\times n}$ and an upper-triangular matrix $T \in \mathbb{C}^{n\times n}$ such that $A = UTU^*$.

```
function SCHURDECOMPOSITION(A)
    if A is 1 × 1 then
        return [1], A
    end if
    (q⃗_1, λ_1) := FINDEIGENVECTOREIGENVALUE(A)
    Q := EXTENDBASIS({q⃗_1}, C^n)        ▷ Extend {q⃗_1} to a basis of C^n using Gram-Schmidt
    Unpack Q := [q⃗_1  Q̃]
    Compute and unpack Q*AQ = [λ_1  a⃗*_12 ; 0⃗_{n-1}  Ã_22]
    (P, T̃) := SCHURDECOMPOSITION(Ã_22)
    U := [q⃗_1  Q̃P]
    T := [λ_1  a⃗*_12 P ; 0⃗_{n-1}  T̃]
    return (U, T)
end function
```

Where:

- Line 3 returns $\begin{bmatrix}1\end{bmatrix}, A$
- Line 5: $(\vec{q}_1, \lambda_1) := \text{FindEigenvectorEigenvalue}(A)$
- Line 6: $Q := \text{ExtendBasis}(\{\vec{q}_1\}, \mathbb{C}^n)$ ▷ Extend $\{\vec{q}_1\}$ to a basis of $\mathbb{C}^n$ using Gram-Schmidt
- Line 7: $Q := \begin{bmatrix} \vec{q}_1 & \widetilde{Q} \end{bmatrix}$
- Line 8: $Q^*AQ = \begin{bmatrix} \lambda_1 & \vec{a}^*_{12} \\ \vec{0}_{n-1} & \widetilde{A}_{22} \end{bmatrix}$
- Line 9: $(P, \widetilde{T}) := \text{SchurDecomposition}(\widetilde{A}_{22})$
- Line 10: $U := \begin{bmatrix} \vec{q}_1 & \widetilde{Q}P \end{bmatrix}$
- Line 11: $T := \begin{bmatrix} \lambda_1 & \vec{a}^*_{12}P \\ \vec{0}_{n-1} & \widetilde{T} \end{bmatrix}$
- Line 12: **return** (U, T)

Spectral Theorem (Diagonalization)

Real symmetric: $A = A^\top \in \mathbb{R}^{n\times n}$ **(Lecture 19)**

$$V^{-1}AV = V^\top AV = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

Hermitian matrix: $A = A^* \in \mathbb{C}^{n\times n}$

$$V^{-1}AV = V^* AV = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

All eigenvalues are real, can be diagonalized by a unitary matrix, and all eigenvectors are orthogonal.
(Proof?)

Singular Value Decomposition

Given $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$, we like to decompose it into a special matrix form: (**Lecture 22**)

$V = [\vec{v}_1, \ldots, \vec{v}_n]$ orthonormal e.v.'s for $A^\top A$ eigenvalues of $A^\top A$ (or $AA^\top$) : $\lambda_1 \geq \cdots \geq \lambda_r > 0 \cdots 0$

$U = [\vec{u}_1, \ldots, \vec{u}_m]$ orthonormal e.v.'s for $AA^\top$ $\Sigma_r = \text{diag}\{\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}\} > 0$

Compact SVD:

$$A = U_r \Sigma_r V_r^\top = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vdots \\ \vec{v}_r^\top \end{bmatrix}$$

Full SVD:

$$A = U \Sigma V^\top = [U_r, U_{m-r}] \begin{bmatrix} \Sigma_r & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(m-r) \times r} & \mathbf{0}_{(m-r) \times (n-r)} \end{bmatrix} \begin{bmatrix} V_r^\top \\ V_{n-r}^\top \end{bmatrix}$$

Singular Value Decomposition

Given $A \in \mathbb{C}^{m \times n}$ with $\operatorname{rank}(A) = r$, we like to decompose it into a special matrix form:

$V = [\vec{v}_1, \ldots, \vec{v}_n]$ orthonormal e.v.'s for A^*A eigenvalues of A^*A (or AA^*) : $\lambda_1 \geq \cdots \geq \lambda_r > 0 \cdots 0$

$U = [\vec{u}_1, \ldots, \vec{u}_m]$ orthonormal e.v.'s for AA^* $\Sigma_r = \operatorname{diag}\{\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}\} > 0$

Compact SVD:

$$A = U_r \Sigma_r V_r^* = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^* \\ \vec{v}_2^* \\ \vdots \\ \vec{v}_r^* \end{bmatrix}$$

Full SVD:

$$A = U \Sigma V^* = [U_r, U_{m-r}] \begin{bmatrix} \Sigma_r & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(m-r) \times r} & \mathbf{0}_{(m-r) \times (n-r)} \end{bmatrix} \begin{bmatrix} V_r^* \\ V_{n-r}^* \end{bmatrix}$$

Moore-Penrose Inverse

$A \in \mathbb{R}^{m \times n}$ **(Lecture 23)**

$$A = U\Sigma V^\top = U \begin{bmatrix} \Sigma_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} V^\top$$

$$A^\dagger = V \begin{bmatrix} \Sigma_r^{-1} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} U^\top = V_r \Sigma_r^{-1} U_r^\top$$

$A \in \mathbb{C}^{m \times n}$

$$A = U\Sigma V^* = U \begin{bmatrix} \Sigma_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} V^*$$

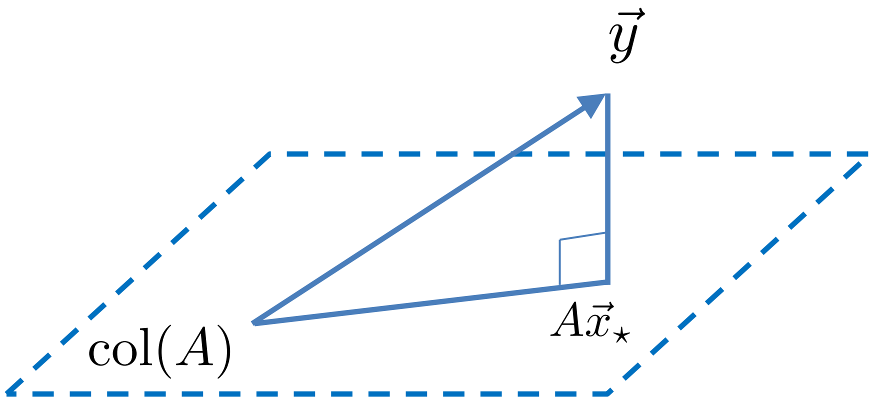
$$A^\dagger = V \begin{bmatrix} \Sigma_r^{-1} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} U^* = V_r \Sigma_r^{-1} U_r^*$$

Solutions to Systems of Linear Equations

$$\vec{y} = A\vec{x}: \;\; \vec{x}_\star = A^\dagger \vec{y}$$

Cases:

1. **square and full rank;**
2. **full column rank (least squares);**
3. **full row rank (least norm);**
4. **general.**



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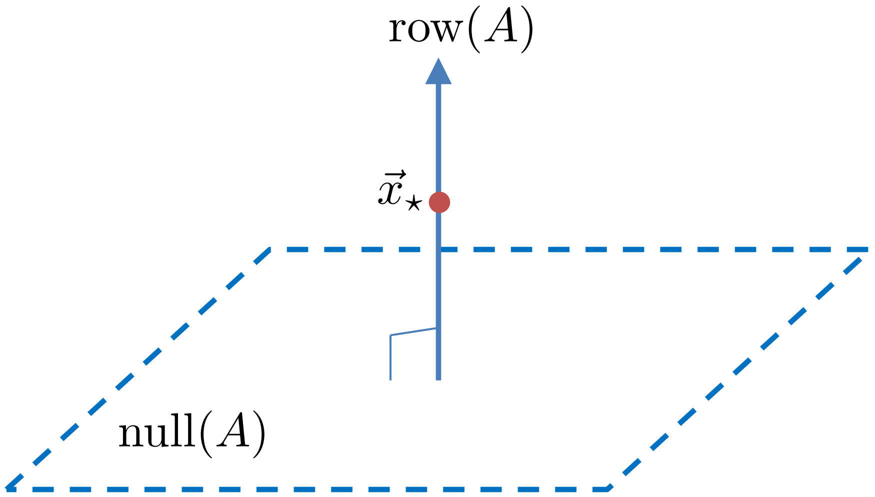

Solutions to Systems of Linear Equations

$$\vec{y} = A\vec{x} : \;\; \vec{x}_\star = A^\dagger \vec{y}$$

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$$A = \quad U \quad \Sigma \quad V^\top$$