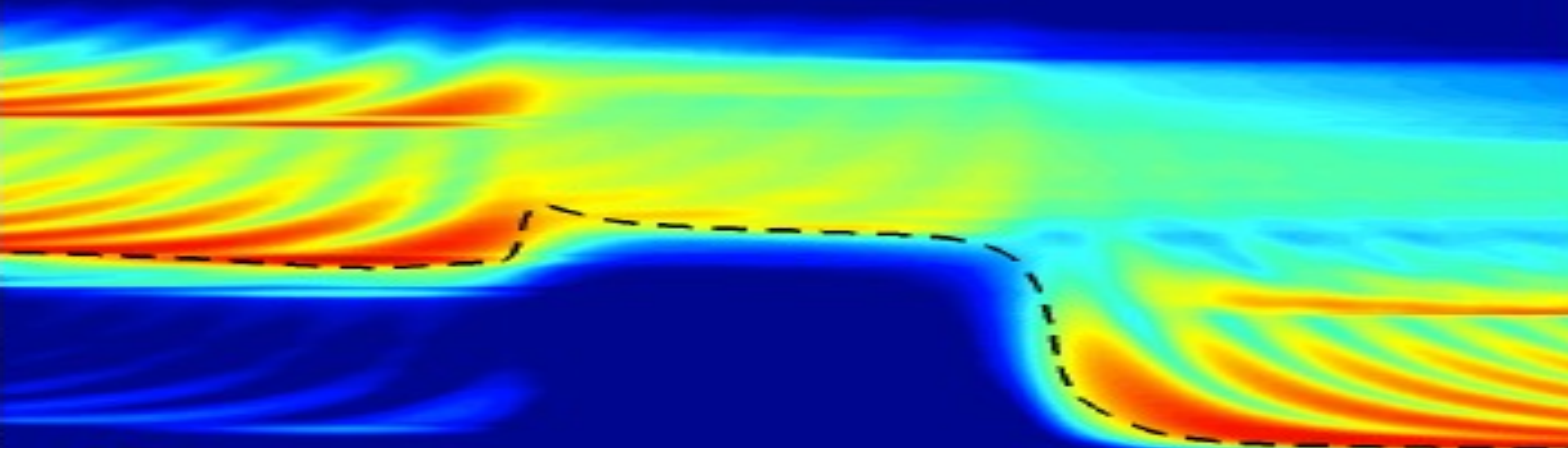

# EECS 16B
# Designing Information Devices and Systems II
# Lecture 27

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# Outline

- Complex Linear Algebra
  - Complex linear vector space
  - Norm and inner product
  - Unitary matrix, Hermitian matrix
  - Gram-Schmidt, Schur Decomposition, SVD
  - Least Squares and Minimum Norm Solutions

$\cos(\omega t + \theta)$

$\mathbb{R}^n$ phasor complex

$A \in \mathbb{R}^{m \times n}$

$p^n(x) = 0 \leftarrow$ real roots

$x^2 + 1 = 0$

# Vector Space

$(\mathbb{R}^n, \mathbb{R})$

$\vec{x}$ $\alpha$

$\alpha \cdot \vec{x}$

A **vector space**: $(\mathbb{V}, \mathbb{F})$ is closed under vector addition and scalar multiplication:

$$\forall \vec{v}_1, \vec{v}_2 \in \mathbb{V}, \text{ and } \forall \alpha, \beta \in \mathbb{F} \qquad \alpha \cdot \vec{v}_1 + \beta \cdot \vec{v}_2 \in \mathbb{V}$$

The addition is associative and commutative; there is an identity/zero vector $\vec{0}$, and every vector has an inverse.

$(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$, $\vec{u}+\vec{v} = \vec{v}+\vec{u}$, $\vec{v}+\vec{0} = \vec{v}$, $\vec{v}+(-\vec{v}) = \vec{0}$

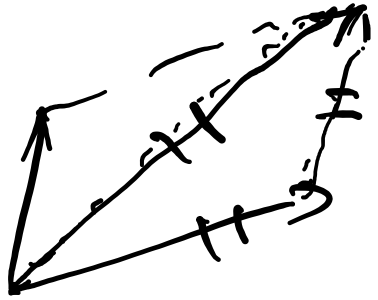
The multiplication is associative, commutative, and distributive; there is an identity scalar **1**.

$(\alpha\beta)\cdot\vec{v} = \alpha(\beta\vec{v})$, $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$. $1\cdot\vec{v} = \vec{v}$

A **norm** $\|\cdot\|$ on the vector space satisfies:

generalization of Euclidean Norm

$$\begin{cases} \|\vec{x}\| \geq 0 \quad \forall \vec{x} \in \mathbb{V} \quad \text{and} \quad \|\vec{x}\| = 0 \Leftrightarrow \vec{x} = \vec{0} \\ \|\alpha\vec{x}\| = |\alpha| \cdot \|\vec{x}\| \quad \forall \vec{x} \in \mathbb{V}, \ \alpha \in \mathbb{F} \\ \|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|, \quad \forall \vec{x}, \vec{y} \in \mathbb{V} \end{cases}$$



# Real versus Complex Vector Space

$(\mathbb{V},\mathbb{F}) = (\mathbb{R}^n,\mathbb{R})$

real vector transpose

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad \vec{x}^\top \doteq [x_1, x_2, \ldots, x_n]$$

T

Inner product: $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i = \vec{y}^\top \vec{x} = \vec{x}^\top \vec{y}$

2-norm: $\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle = \vec{x}^\top \vec{x} = \sum_{i=1}^{n} x_i^2$

$$\|\vec{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$\alpha_i + \beta_i j \in \mathbb{C}$

$(\mathbb{V},\mathbb{F}) = (\mathbb{C}^n,\mathbb{C})$ **(note2j)**

complex conjugate transpose

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n, \quad \vec{x}^* \doteq (\bar{\vec{x}})^\top = [\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n]$$

$*$ $\quad \alpha_i - \beta_i j$

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i \bar{y}_i = \vec{y}^* \vec{x} \quad \left( = \overline{\vec{x}^* \vec{y}} = \overline{\langle \vec{y}, \vec{x} \rangle} \right)$$

$$\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle = \vec{x}^* \vec{x} = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2$$

$$\|\vec{x}\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

# Complex Vector Norm and Inner Product

$x = \alpha + j\beta \quad \bar{x} = \alpha - j\beta$

$\vec{x} = \begin{bmatrix} 1 \\ 2j \end{bmatrix} \in \mathbb{C}^2$

$\|\vec{x}\|^2 = 1^2 + (\cancel{2j})^2 = 1 - 4 = -3$

$\|\vec{x}\|^2 = 1^2 + |2j|^2 = 1 + 4 = 5$

$\overset{\|}{\langle \vec{x}, \vec{x} \rangle} = \vec{x}^* \vec{x} = [1, -2j] \begin{bmatrix} 1 \\ 2j \end{bmatrix} = 1 + 4.$

$\vec{x} = \begin{bmatrix} 1+2j \\ 2+2j \end{bmatrix}$

$\|\vec{x}\|^2 = |x_1|^2 + |x_2|^2 = (1^2 + 2^2) + (2^2 + 2^2)$

$= 5 + 8 = 13$

$\vec{x}_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} j \\ 1 \end{bmatrix}$

$\langle \vec{x}_1, \vec{x}_2 \rangle = \vec{x}_2^* \vec{x}_1 = [-j, 1] \begin{bmatrix} 1 \\ j \end{bmatrix} = 0.$

$\vec{x}_1 \perp \vec{x}_2$

$\vec{x}_2^T \vec{x}_1 = 2j$

# Real versus Complex Matrix

$$(\mathbb{V}, \mathbb{F}) = (\mathbb{R}^n, \mathbb{R})$$

real matrix transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m,n-1} & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

$$\downarrow$$

$$A^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{1n} & \cdots & a_{m-1,n} & a_{mn} \end{bmatrix} \in \mathbb{R}^{n\times m}$$

$$(\mathbb{V}, \mathbb{F}) = (\mathbb{C}^n, \mathbb{C})$$

complex conjugate transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m,n-1} & a_{mn} \end{bmatrix} \in \mathbb{C}^{m\times n}$$

$$\downarrow$$

$$\overline{a_{ij}}$$

$$A^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{m1} \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ \bar{a}_{1n} & \cdots & \bar{a}_{m-1,n} & \bar{a}_{mn} \end{bmatrix} \in \mathbb{C}^{n\times m}$$

# Complex Matrices

Algebraic manipulations, row, column, null space, rank, inverse, eigenvectors and eigenvalues are all similar to those of real matrices.

$$\vec{y} = \underline{A}\vec{x} \qquad A \in \mathbb{C}^{m\times n} \qquad \vec{x} \in \mathbb{C}^{n}, \quad \vec{y} \in \mathbb{C}^{m}$$

$$A\cdot B, \quad A+B, \quad \alpha \cdot A, \quad \alpha \in \mathbb{C}$$

$$A\vec{x} = \vec{0} \in \mathbb{C}^{m} \qquad \vec{x} \in \mathbb{C}^{n} \qquad \vec{x} \in Null(A)$$

$$A \in \mathbb{C}^{n\times n}, \quad \det(A) \neq 0 \qquad A^{-1}A = AA^{-1} = I_{n\times n}$$

$$A\vec{v}_i = \lambda_i \vec{v}_i, \quad \text{eigenvectors / eigenvalues}, \quad \underline{\det(\lambda I - A) = 0}$$

# Real versus Complex Matrices

$$(\mathbb{V}, \mathbb{F}) = (\mathbb{R}^n, \mathbb{R})$$

Orthogonal Matrix: $Q = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n] \in \mathbb{R}^{n\times n}$

$$\vec{q}_i^{\top} \vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \quad \text{(orthogonal)} \\ 1 & \text{if } i = j \quad \text{(normalized)} \end{cases}$$

$$Q^\top Q = I = QQ^\top$$

$$Q^{-1} = Q^T$$

rotation

$$(\mathbb{V}, \mathbb{F}) = (\mathbb{C}^n, \mathbb{C})$$

Unitary Matrix: $Q = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n] \in \mathbb{C}^{n\times n}$

$$\vec{q}_i^{*} \vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \quad \text{(orthogonal)} \\ 1 & \text{if } i = j \quad \text{(normalized)} \end{cases}$$

$$Q^* Q = I = QQ^*$$

$$Q^{-1} = Q^*$$

# Gram-Schimdt Orthonormalization (QR)

$D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] \in \mathbb{R}^{n \times k}$ **(Lecture 17)**

$\vec{q}_1 \cdots \vec{q}_r$

QR: $[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$

$D = QR$

$\vec{y} = D\vec{x}$

$= QR\vec{x}$

$Q^T\vec{y} = R\vec{x}$

step 1.

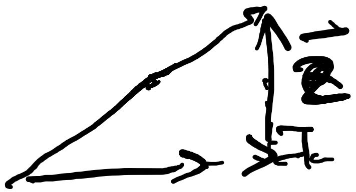
Gram-Schimdt: $\vec{z}_1 = \vec{d}_1$

step 3 $\vec{z}_2 = \vec{d}_2 - (\vec{d}_2^\top \vec{q}_1)\vec{q}_1$

$\vec{z}_3 = \vec{d}_3 - (\vec{d}_3^\top \vec{q}_1)\vec{q}_1 - (\vec{d}_3^\top \vec{q}_2)\vec{q}_2$

$\vdots$

$\vec{z}_k = \vec{d}_k - \sum_{j=1}^{k-1} (\vec{d}_k^\top \vec{q}_j)\vec{q}_j$



step 2.

$\vec{q}_1 = \vec{z}_1 / \|\vec{z}_1\|$

$\vec{q}_2 = \vec{z}_2 / \|\vec{z}_2\|$

$\vec{q}_3 = \vec{z}_3 / \|\vec{z}_3\|$

$\vdots$

$\vec{q}_k = \vec{z}_k / \|\vec{z}_k\|$

$\boxed{\vec{d}_{r+1} - Q_r Q_r^T \vec{d}_{r+1}}$

proj$(\vec{d}_{r+1})$

on col$(Q_r)$

# Gram-Schimdt Orthonormalization (QR)

$$D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] \ \in \mathbb{C}^{n\times k}$$

QR:

$$[\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_k] = \underline{[\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k]} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & \cdots & r_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{kk} \end{bmatrix}$$

Gram-Schimdt:

$$\vec{z}_1 = \vec{d}_1 \qquad \vec{q}_1 = \vec{z}_1/\|\vec{z}_1\|$$

$$\vec{z}_2 = \vec{d}_2 - \langle \vec{d}_2, \vec{q}_1\rangle \vec{q}_1 \qquad \vec{q}_2 = \vec{z}_2/\|\vec{z}_2\|$$

$$\vec{z}_3 = \vec{d}_3 - \langle \vec{d}_3, \vec{q}_1\rangle \vec{q}_1 - \langle \vec{d}_3, \vec{q}_2\rangle \vec{q}_2 \qquad \vec{q}_3 = \vec{z}_3/\|\vec{z}_3\|$$

$$\vdots \qquad \vdots$$

$$\vec{z}_k = \vec{d}_k - \sum_{j=1}^{k-1} \underline{\langle \vec{d}_k, \vec{q}_j\rangle \vec{q}_j} \qquad \vec{q}_k = \vec{z}_k/\|\vec{z}_k\|$$

$$Q_r^* = \begin{bmatrix} \vec{q}_1^{\,*} \\ \vec{q}_2^{\,*} \\ \vdots \\ \vec{q}_r^{\,*} \end{bmatrix}$$

$$\Leftarrow \quad \vec{z}_{r+1} = \vec{d}_{r+1} - \underline{Q_r Q_r^* \vec{d}_{r+1}}$$

# Schur Decomposition (Upper Triangularization)

$A \in \mathbb{R}^{n \times n}$ **(Lecture 18)**

$$T = U^{-1}AU = U^{\top}AU = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

$$A = U T U^{T} = U T U^{-1}$$

$$A \vec{q}_1 = \lambda_1 \vec{q}_1$$

$$\vec{x}[i+1] = A \vec{x}[i] \quad ①$$

$$U^{T}\vec{x}[i+1] = T\, U^{T}\vec{x}[i]$$

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**Algorithm 10** Real Schur Decomposition

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**Input:** A square matrix $A \in \mathbb{R}^{n \times n}$ with real eigenvalues.

**Output:** An orthonormal matrix $U \in \mathbb{R}^{n \times n}$ and an upper-triangular matrix $T \in \mathbb{R}^{n \times n}$ such that $A = UTU^{\top}$.

**function** RealSchurDecomposition($A$)
  **if** $A$ is $1 \times 1$ **then**
    **return** $\begin{bmatrix} 1 \end{bmatrix}, A$
  **end if**
  $(\vec{q}_1, \lambda_1) :=$ FindEigenvectorEigenvalue($A$)
  $Q :=$ ExtendBasis($\{\vec{q}_1\}, \mathbb{R}^n$)  ▷ Extend $\{\vec{q}_1\}$ to a basis of $\mathbb{R}^n$ using Gram-Schmidt; see Note 13
  Unpack $Q := \begin{bmatrix} \vec{q}_1 & \widetilde{Q} \end{bmatrix}$ ← G.S.
  Compute and unpack $Q^{\top}AQ = \begin{bmatrix} \lambda_1 & \vec{a}_{12}^{\top} \\ \vec{0}_{n-1} & \widetilde{A}_{22} \end{bmatrix}$
  $(P, \widetilde{T}) :=$ RealSchurDecomposition($\widetilde{A}_{22}$)
  $U := \begin{bmatrix} \vec{q}_1 & \widetilde{Q}P \end{bmatrix}$
  $T := \begin{bmatrix} \lambda_1 & \vec{a}_{12}^{\top}P \\ \vec{0}_{n-1} & \widetilde{T} \end{bmatrix}$
  **return** $(U, T)$
**end function**

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$$\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} \qquad Q^{T} \lambda \vec{q}_1$$

$$\begin{bmatrix} \vec{q}_1^{\,T} \\ \vdots \end{bmatrix} \vec{q}_1 \cdot \lambda$$

$$\vec{z}[i+1] = T\, \vec{z}[i] \quad ②$$

# Schur Decomposition (Upper Triangularization)

$A \in \mathbb{C}^{n \times n}$

$$T = U^{-1}AU = U^*AU = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

$A = UTU^*$

$T = U^*AU$

**Algorithm 64** Schur Decomposition

**Input:** A square matrix $A \in \mathbb{C}^{n \times n}$.

**Output:** A unitary matrix $U \in \mathbb{C}^{n \times n}$ and an upper-triangular matrix $T \in \mathbb{C}^{n \times n}$ such that $A = UTU^*$.

**function** SCHURDECOMPOSITION($A$)
  **if** $A$ is $1 \times 1$ **then**
    **return** $\begin{bmatrix} 1 \end{bmatrix}, A$
  **end if**
  $(\vec{q}_1, \lambda_1) :=$ FINDEIGENVECTOREIGENVALUE($A$)
  $Q :=$ EXTENDBASIS($\{\vec{q}_1\}, \mathbb{C}^n$) ▷ Extend $\{\vec{q}_1\}$ to a basis of $\mathbb{C}^n$ using Gram-Schmidt
  Unpack $Q := \begin{bmatrix} \vec{q}_1 & \widetilde{Q} \end{bmatrix}$
  Compute and unpack $Q^*AQ = \begin{bmatrix} \lambda_1 & \vec{a}_{12}^* \\ \vec{0}_{n-1} & \widetilde{A}_{22} \end{bmatrix}$
  $(P, \widetilde{T}) :=$ SCHURDECOMPOSITION($\widetilde{A}_{22}$)
  $U := \begin{bmatrix} \vec{q}_1 & \widetilde{Q}P \end{bmatrix}$
  $T := \begin{bmatrix} \lambda_1 & \vec{a}_{12}^* P \\ \vec{0}_{n-1} & \widetilde{T} \end{bmatrix}$
  **return** $(U, T)$
**end function**

# Spectral Theorem (Diagonalization)

Real symmetric: $A = A^\top \in \mathbb{R}^{n\times n}$ **(Lecture 19)**

$$V^{-1}AV = V^\top AV = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

Hermitian matrix: $A = A^* \in \mathbb{C}^{n\times n}$

$$V^{-1}AV = V^* AV = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \in$$

All eigenvalues are real, can be diagonalized by a unitary matrix, and all eigenvectors are orthogonal. (Proof?)

$A = VTV^T \qquad T = \Lambda$

$A^T = VT^TV^T$

$A \in \mathbb{R}^{m\times n} \in \mathbb{C}^{m\times n}$

$A^TA \quad AA^T \quad A^*A \quad AA^*$

# Singular Value Decomposition

Given $A \in \mathbb{R}^{m\times n}$ with $\mathrm{rank}(A) = r$, we like to decompose it into a special matrix form: (**Lecture 22**)

$V = [\vec{v}_1, \ldots, \vec{v}_n]$ orthonormal e.v.'s for $A^\top A$ eigenvalues of $A^\top A$ (or $AA^\top$) : $\lambda_1 \geq \cdots \geq \lambda_r > 0 \cdots 0$

$U = [\vec{u}_1, \ldots, \vec{u}_m]$ orthonormal e.v.'s for $AA^\top$ $\Sigma_r = \mathrm{diag}\{\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}\} > 0$

$\vec{v}^\top A^\top A \vec{v} = \vec{v}^\top \lambda \vec{v} \Rightarrow \lambda \geq 0$

Compact SVD: $A = U_r \Sigma_r V_r^\top = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vdots \\ \vec{v}_r^\top \end{bmatrix}$

Full SVD: $A = U \Sigma V^\top = [U_r, U_{m-r}] \begin{bmatrix} \Sigma_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} \begin{bmatrix} V_r^\top \\ V_{n-r}^\top \end{bmatrix}$

# Singular Value Decomposition

Given $A \in \mathbb{C}^{m\times n}$ with $\mathrm{rank}(A) = r$, we like to decompose it into a special matrix form:

$V = [\vec{v}_1, \ldots, \vec{v}_n]$ orthonormal e.v.'s for $A^*A$ eigenvalues of $A^*A$ (or $AA^*$) : $\lambda_1 \geq \cdots \geq \lambda_r > 0 \cdots 0$

$U = [\vec{u}_1, \ldots, \vec{u}_m]$ orthonormal e.v.'s for $AA^*$ $\Sigma_r = \mathrm{diag}\{\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}\} > 0$

Compact SVD:

$$A = U_r \Sigma_r V_r^* = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^* \\ \vec{v}_2^* \\ \vdots \\ \vec{v}_r^* \end{bmatrix}$$

Full SVD:

$$A = U\Sigma V^* = [U_r, U_{m-r}] \begin{bmatrix} \Sigma_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} \begin{bmatrix} V_r^* \\ V_{n-r}^* \end{bmatrix}$$

# Moore-Penrose Inverse

$A \in \mathbb{R}^{m\times n}$ **(Lecture 23)**

$$A = U\Sigma V^\top = U \begin{bmatrix} \Sigma_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} V^\top$$

$$A^\dagger = V \begin{bmatrix} \Sigma_r^{-1} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} U^\top = V_r \Sigma_r^{-1} U_r^\top$$

$A \in \mathbb{C}^{m\times n}$

$$A = U\Sigma V^* = U \begin{bmatrix} \Sigma_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} V^*$$

$$A^\dagger = V \begin{bmatrix} \Sigma_r^{-1} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} U^* = V_r \Sigma_r^{-1} U_r^*$$

$$A_{n\times n} \rightarrow A^{-1} \qquad y = Ax \Rightarrow x = \underline{A^{-1}}\, y$$

$$A^\dagger \rightarrow A^{-1}$$

# Solutions to Systems of Linear Equations

$$\vec{y} = A\vec{x}: \quad \vec{x}_\star = A^\dagger \vec{y}$$

**Cases:**
1. **square and full rank;**
2. **full column rank (least squares);**
3. **full row rank (least norm);**
4. **general.**

$\vec{y} = [\ \ ]\vec{x} \leftarrow$ unknowns $(A, B)$

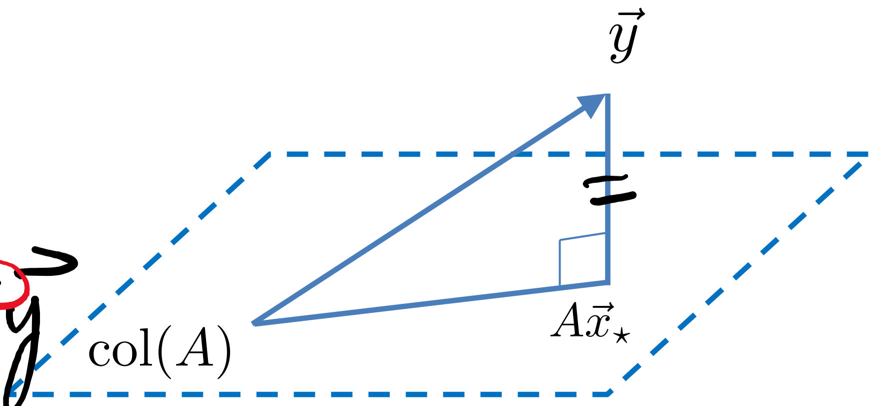
$n \qquad n, u$

$$\vec{y} - A\vec{x}_* \perp \mathrm{col}(A)$$

$$\min \|\vec{y} - A\vec{x}\|_2^2$$

$$A^T(\vec{y} - A\vec{x}_*) = 0$$

$$A^T\vec{y} = (A^TA)\vec{x}_* \Rightarrow \vec{x}_* = \overbrace{(A^TA)^{-1}A^T}^{A^\dagger}\vec{y}$$



# Solutions to Systems of Linear Equations

$$\vec{y} = A\vec{x}: \quad \vec{x}_\star = A^\dagger \vec{y}$$

**Cases:**
1. square and full rank;
2. full column rank (least squares);
3. full row rank (least norm);
4. general.

$$\vec{y} = \boxed{A}\,\vec{x}$$

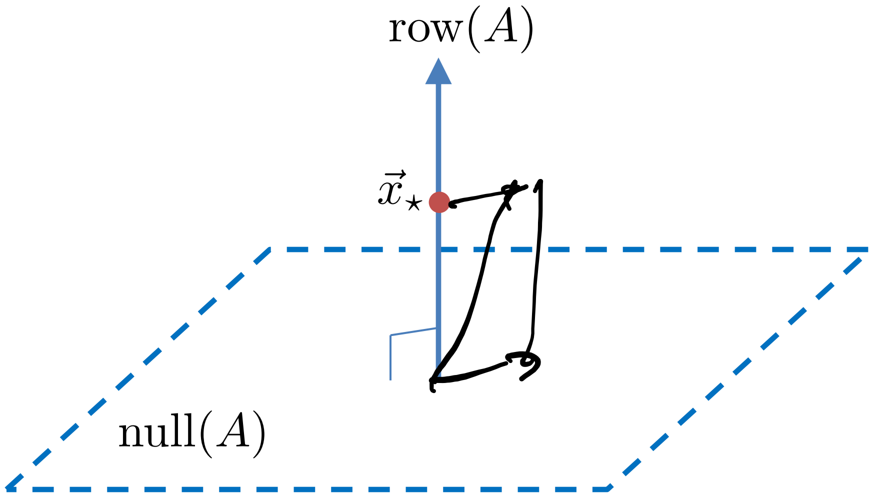
unknown: $(\vec{u})$

$$\vec{x}_\star \in \text{row}(A)$$

$$\vec{x}_\star = A^T \vec{w}$$

$$\vec{y} = AA^T\vec{w} \Rightarrow \vec{w} = (AA^T)^{-1}\vec{y}$$

$$\vec{x}_\star = A^T(AA^T)^{-1}\vec{y}$$



# Solutions to Systems of Linear Equations

$$\vec{y} = A\vec{x} : \ \ \vec{x}_\star = A^\dagger \vec{y}$$

**Cases:**

1. **square and full rank;**
2. **full column rank (least squares);**
3. **full row rank (least norm);**
4. **general.**

$$A = \qquad U \qquad \Sigma \qquad V^\top$$