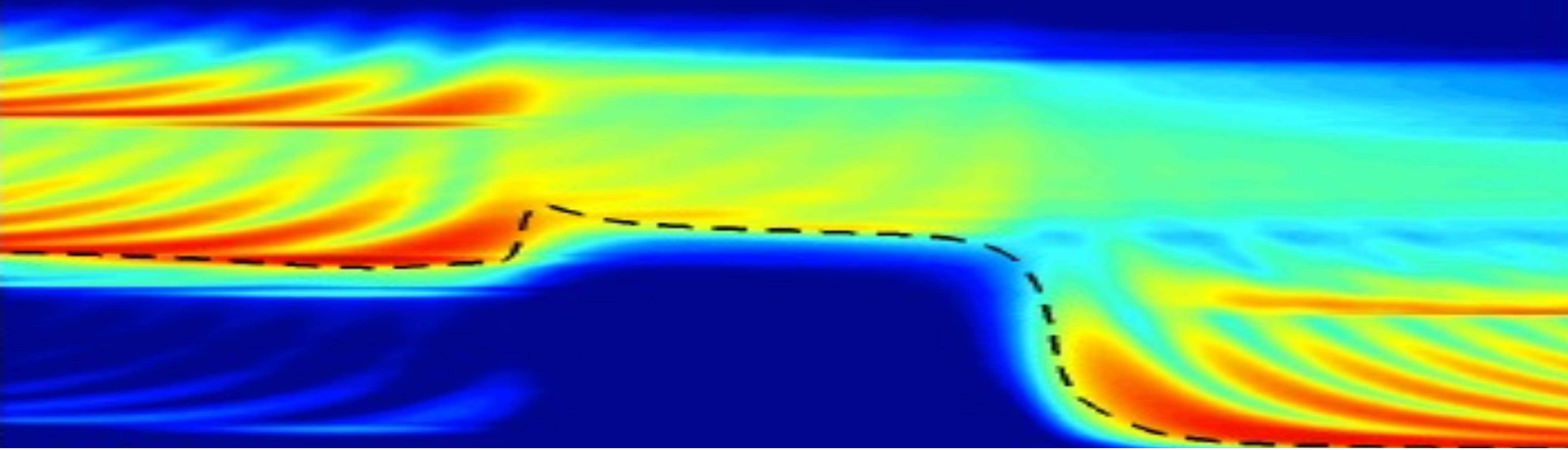


EECS 16B
Designing Information Devices and Systems II
Lecture 28

Prof. Yi Ma

Department of Electrical Engineering and Computer Sciences, UC Berkeley,
yima@eecs.berkeley.edu

Outline

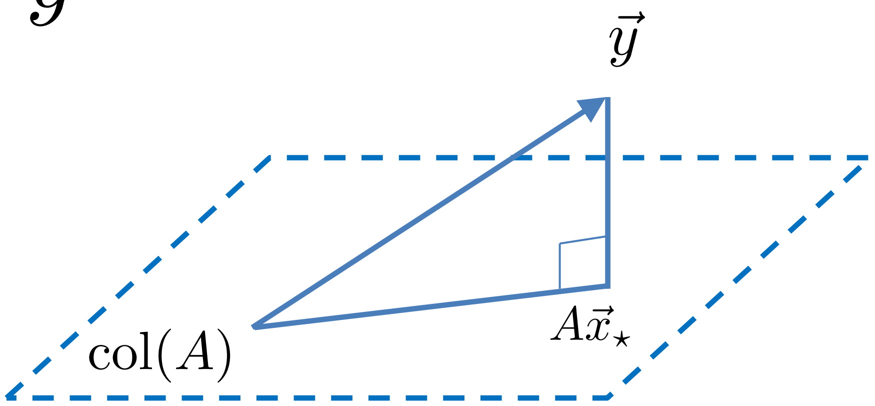
- Final Review (part two)
  - Solutions to Linear Equations
  - System Discretization & Identification
  - System Stability
  - System Controllability
  - Minimum Energy Control
  - Principal Component Analysis (PCA)

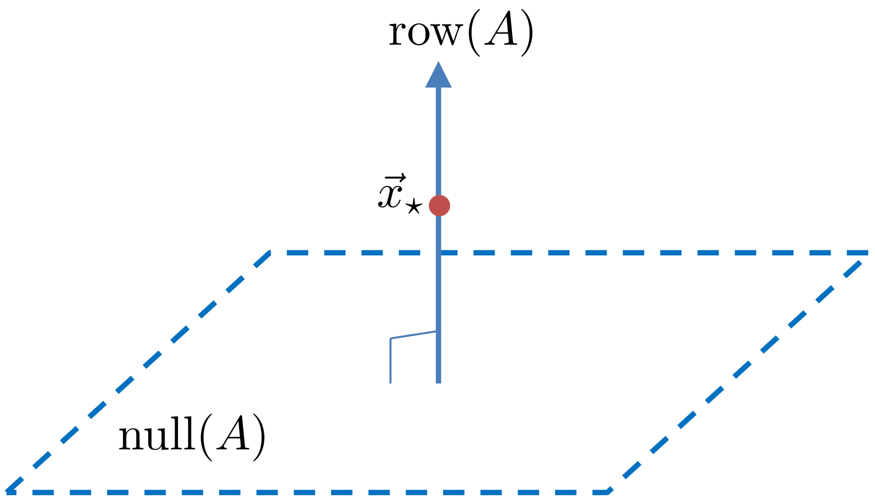
Solutions to Systems of Linear Equations

$$\vec{y} = A\vec{x} : \;\; \vec{x}_\star = A^\dagger \vec{y}$$

Cases:

1. **square and full rank (inverse);**
2. **full column rank (least squares, system identification);**
3. **full row rank (least norm, minimum energy control);**
4. **general cases.**



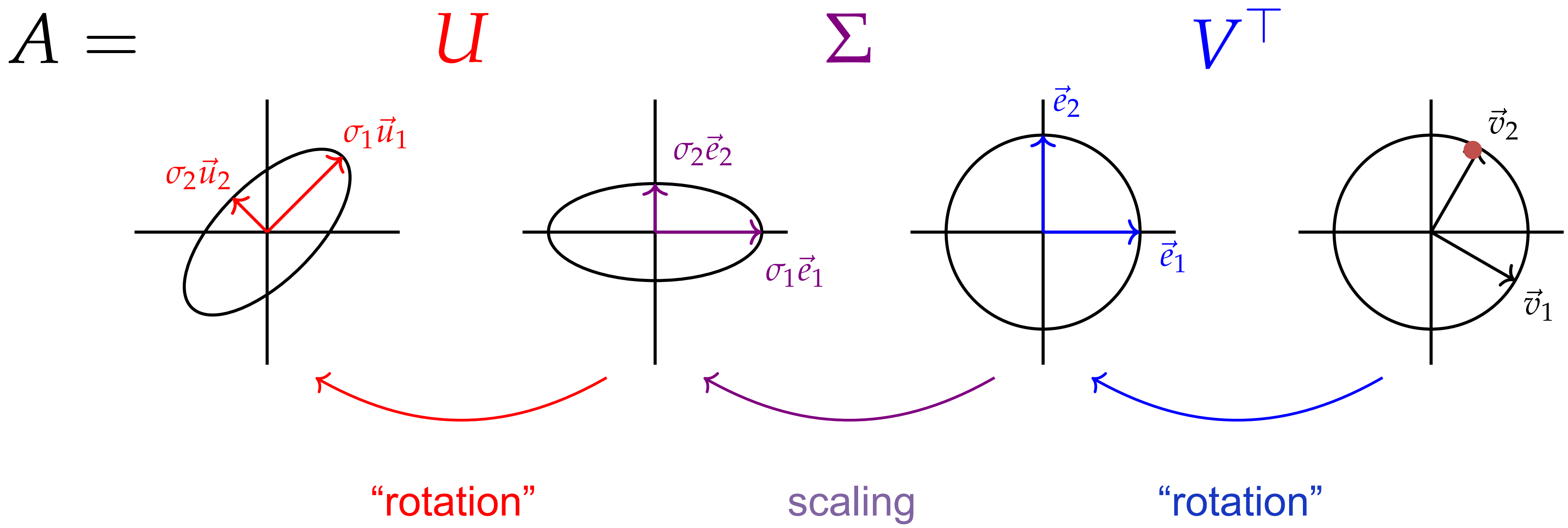


Solutions to Systems of Linear Equations

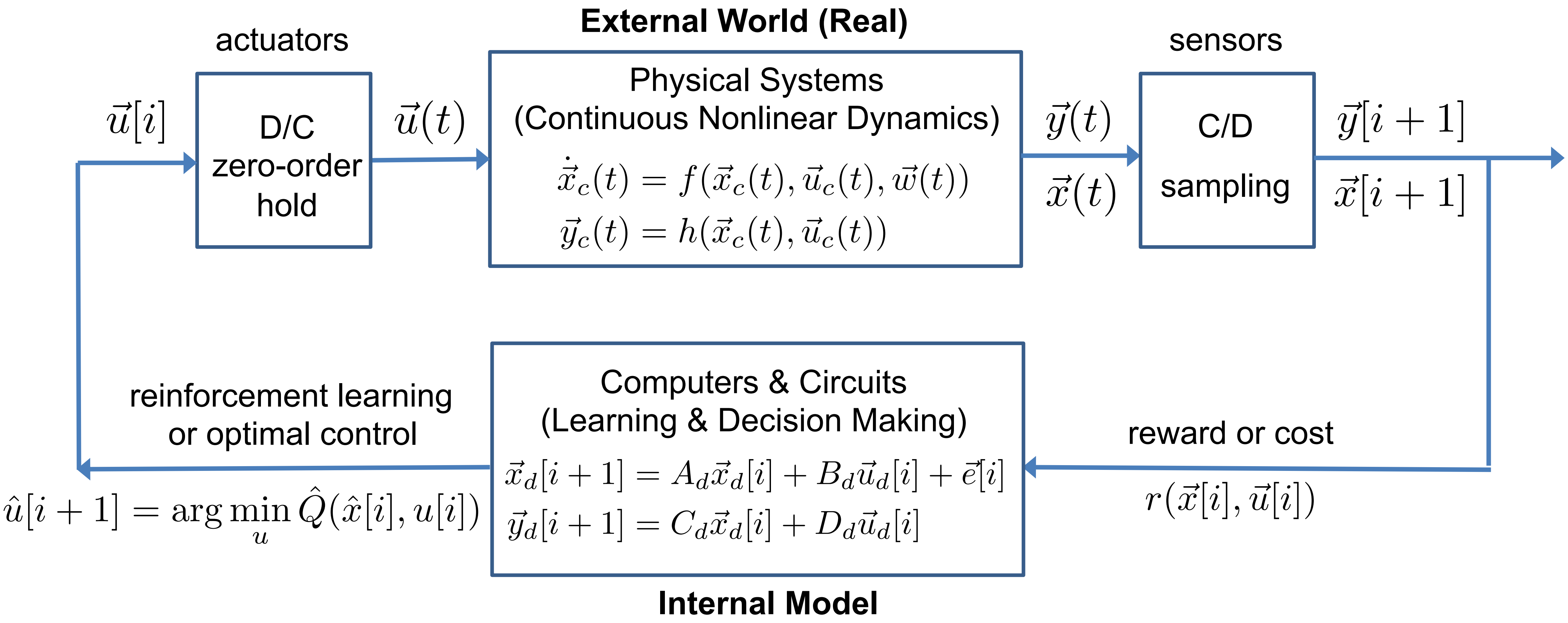
$$\vec{y} = A\vec{x} : \;\; \vec{x}_\star = A^\dagger \vec{y}$$

Cases:

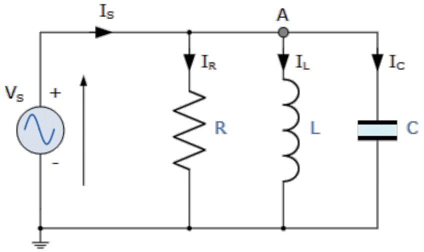
1. **square and full rank (inverse);**
2. **full column rank (least squares, system identification);**
3. **full row rank (least norm, minimum energy control);**
4. **general cases: pseudo inverse, PCA etc.**

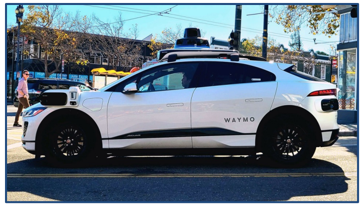


System Modeling, Analysis, & Control



System Modeling





mathematical modeling
from first principles

$$\dot{\vec{x}}_c(t) = f(\vec{x}_c(t), \vec{u}_c(t), \vec{w}(t))$$
$$\vec{y}_c(t) = h(\vec{x}_c(t), \vec{u}_c(t))$$

approximation
& linearization

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{n}(t)$$
$$\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$$

discretization
& digitization

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] + \vec{e}[i]$$
$$\vec{y}_d[i+1] = C_d\vec{x}_d[i] + D_d\vec{u}_d[i]$$

Discretization (Lecture 12)

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$$

$$\vec{x}(t) = e^{A(t-t_0)}\vec{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B\vec{u}(\tau)d\tau$$

$$\vec{x}_d[i+1] = e^{A\Delta}\vec{x}_d[i] + \int_{i\Delta}^{(i+1)\Delta} e^{A(t-\tau)}Bd\tau\vec{u}_d[i]$$

$$A_d = e^{A\Delta}$$

$$B_d = (e^{A\Delta} - I)A^{-1}B$$

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i]$$

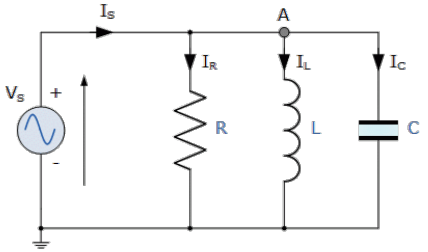
System Modeling: Identification

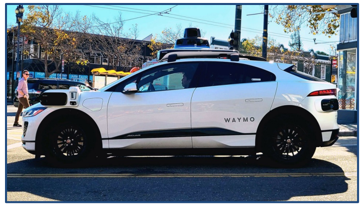
Identification: **(Lecture 13)** $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$

From observations: $\vec{u}[0], \vec{u}[1], \ldots, \vec{u}[l], \ldots$

$\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[l], \ldots$

System Analysis





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discretization
& digitization

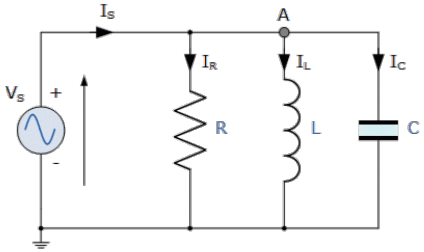
$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] + \vec{e}[i]$$
$$\vec{y}_d[i+1] = C_d\vec{x}_d[i] + D_d\vec{u}_d[i]$$

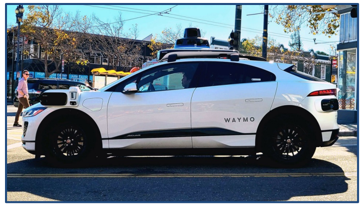
Stability Criteria (Lecture 14)

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$$

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i]$$

System Control





mathematical modeling
from first principles

$$\dot{\vec{x}}_c(t) = f(\vec{x}_c(t), \vec{u}_c(t), \vec{w}(t))$$
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discretization
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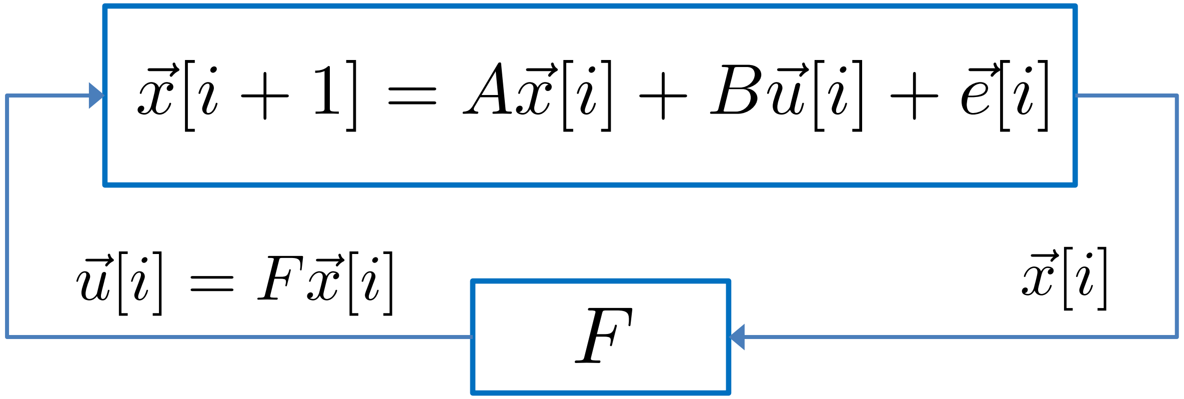
$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] + \vec{e}[i]$$
$$\vec{y}_d[i+1] = C_d\vec{x}_d[i] + D_d\vec{u}_d[i]$$

Controllability (Lecture 15)



System Control

Controllable Canonical Form: (Lecture 16)

$$\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \quad \in \mathbb{R}^n$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$F = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \end{bmatrix}$$

$$\det(\lambda I - A) =$$

$$\lambda^n - a_n\lambda^{n-1} - a_{n-1}\lambda a^{n-2} - \cdots - a_2\lambda - a_1$$

System Control

Design control input to steer the state of a controllable system:

$\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ $\quad$ $\mathcal{C} \doteq [A^{n-1}B \mid \cdots \mid AB \mid B] \in \mathbb{R}^{n\times n}$ is invertible.

$$\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots \mid AB \mid B] \in \mathbb{R}^{n\times \ell}$$

System State Estimation

Estimate the state of the system from observable outputs:

$$\vec{x}_d[i+1] = A\vec{x}_d[i] + B\vec{u}_d[i]$$
$$\vec{y}_d[i+1] = C\vec{x}_d[i] + D\vec{u}_d[i]$$

SVD, Low-Rank Approximation, PCA

$$A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n] \in \mathbb{R}^{m \times n} \qquad A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top = U_r \Sigma_r V_r^\top$$

Low-rank Approximation: $\min_{B \in \mathbb{R}^{m \times n}} \|A - B\|_F^2 \quad \text{subject to} \quad \text{rank}(B) = \ell$ **(Lecture 24)**

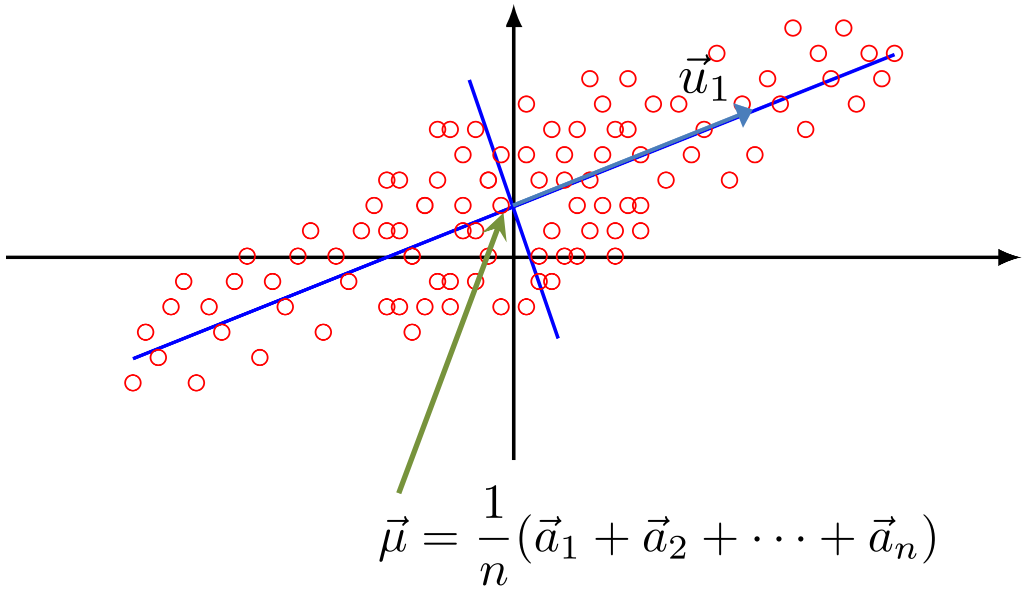
$$A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n] = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top = \sum_{i=1}^{\ell} \sigma_i \vec{u}_i \vec{v}_i^\top + \sum_{i=\ell+1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top$$

SVD, Low-Rank Approximation, PCA

$$A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n] \in \mathbb{R}^{m \times n} \qquad \vec{\mu} = \frac{1}{n}(\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n) = \mathbf{0}$$

Principal Component: **(Lecture 24)**

find a normal vector $\|\vec{u}\|_2 = 1$ such that $\max_{\vec{u}} \|\vec{u}^\top A\|_2^2 = \|\vec{u}\vec{u}^\top A\|_2^2$.



SVD, Low-Rank Approximation, PCA

$$A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n] \in \mathbb{R}^{m \times n} \qquad \vec{\mu} = \frac{1}{n}(\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n) = \mathbf{0}$$

Principal Components: **(Lecture 24)**

Find projection: $\max_{U_\ell} \|U_\ell U_\ell^\top A\|_F^2 \quad \Leftrightarrow \min_{U_\ell} \|A - U_\ell U_\ell^\top A\|_F^2 \quad \Leftrightarrow \quad \min_{U_{m-\ell}} \|U_{m-\ell} U_{m-\ell}^\top A\|_F^2$