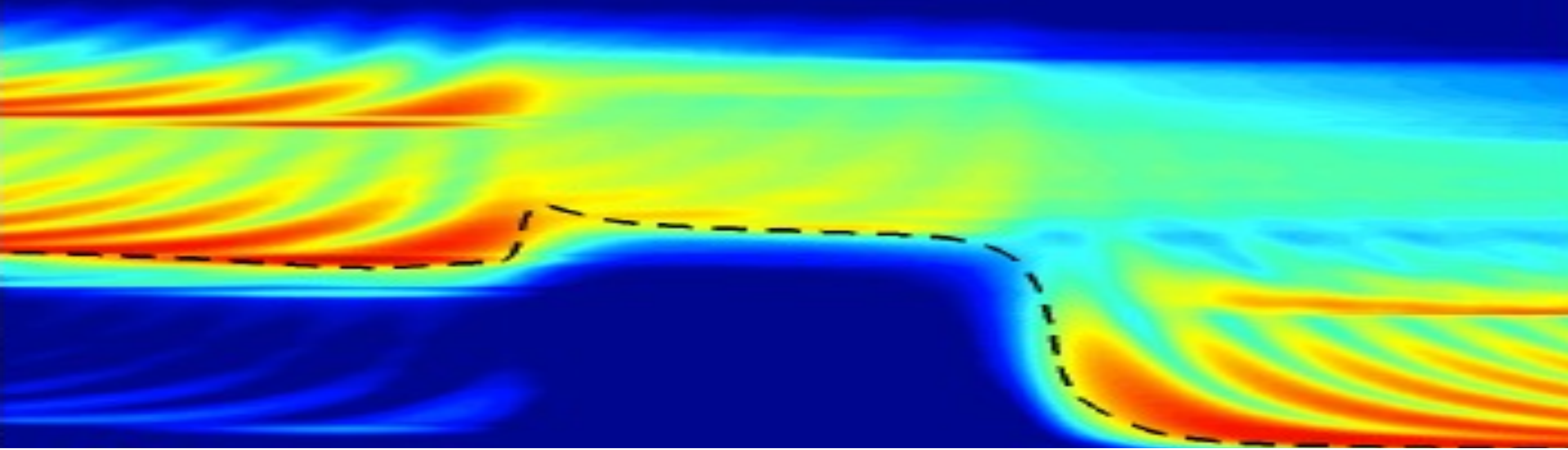

EECS 16B
Designing Information Devices and Systems II
Lecture 28

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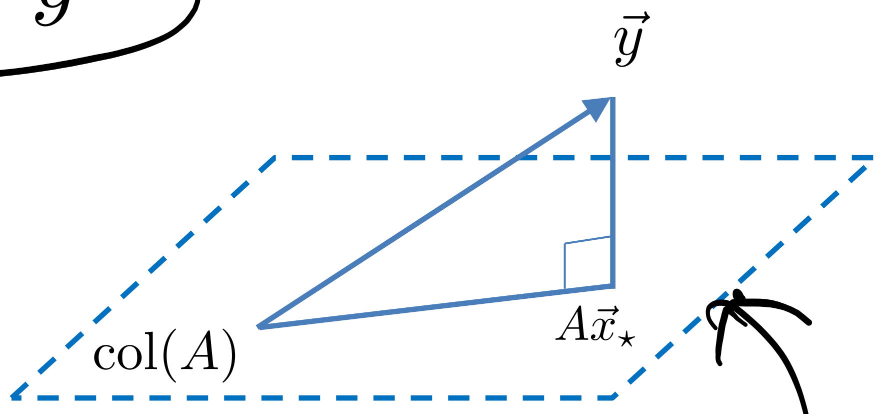
Outline

- Final Review (part two)
  - Solutions to Linear Equations $y = Ax$
  - System Discretization & Identification
  - System Stability
  - System Controllability
  - Minimum Energy Control
  - Principal Component Analysis (PCA)

Solutions to Systems of Linear Equations

$$\vec{y} = A\vec{x}: \quad \vec{x}_\star = A^\dagger \vec{y}$$

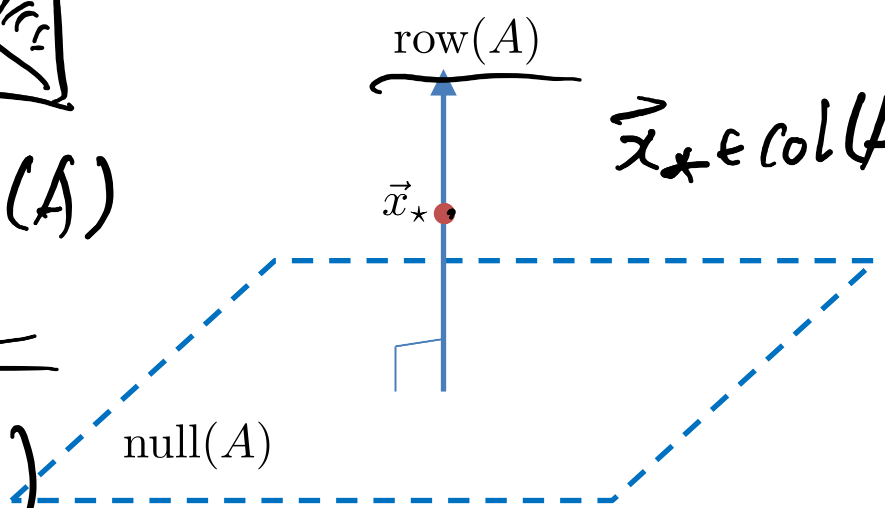
Cases:
1. **square and full rank (inverse);**
2. **full column rank (least squares, system identification);**
3. **full row rank (least norm, minimum energy control);**
4. **general cases.**



① A - square invertible. $\vec{x} = A^{-1}\vec{y}$

$\vec{y} = QR\vec{x} \qquad Q^T\vec{y} = R\vec{x}$

② $\boxed{A}$ $\min \|\vec{y} - A\vec{x}\|_2^2 \qquad \vec{y} - A\vec{x}_* \perp \mathrm{col}(A)$



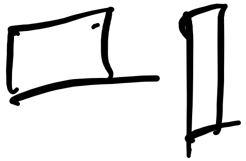
$\vec{x}_* \in \mathrm{col}(A^T)$

③ $\boxed{A}$ $\min \|\vec{x}\|_2^2$ s.t. $\vec{y} = A\vec{x}$

$\vec{x}_* \in \mathrm{col}(A^T)$

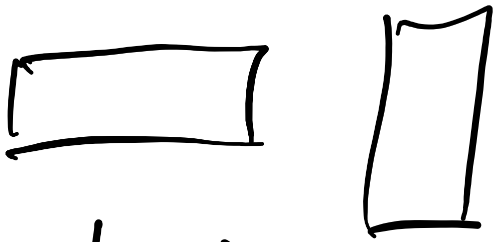
$\vec{x}_* \in \mathrm{col}(A^T)$ $\qquad$ $\vec{x}_* = \underline{A^T \vec{w}}$

$\vec{y} = A A^T \vec{w}$ $\qquad$ $\vec{w} = (\underline{A A^T})^{-1} \vec{y}$



Solutions to Systems of Linear Equations

$$\vec{y} = A\vec{x} : \quad \underline{\vec{x}_\star = A^\dagger \vec{y}}$$



rank (r)

$<$ min (m, n)

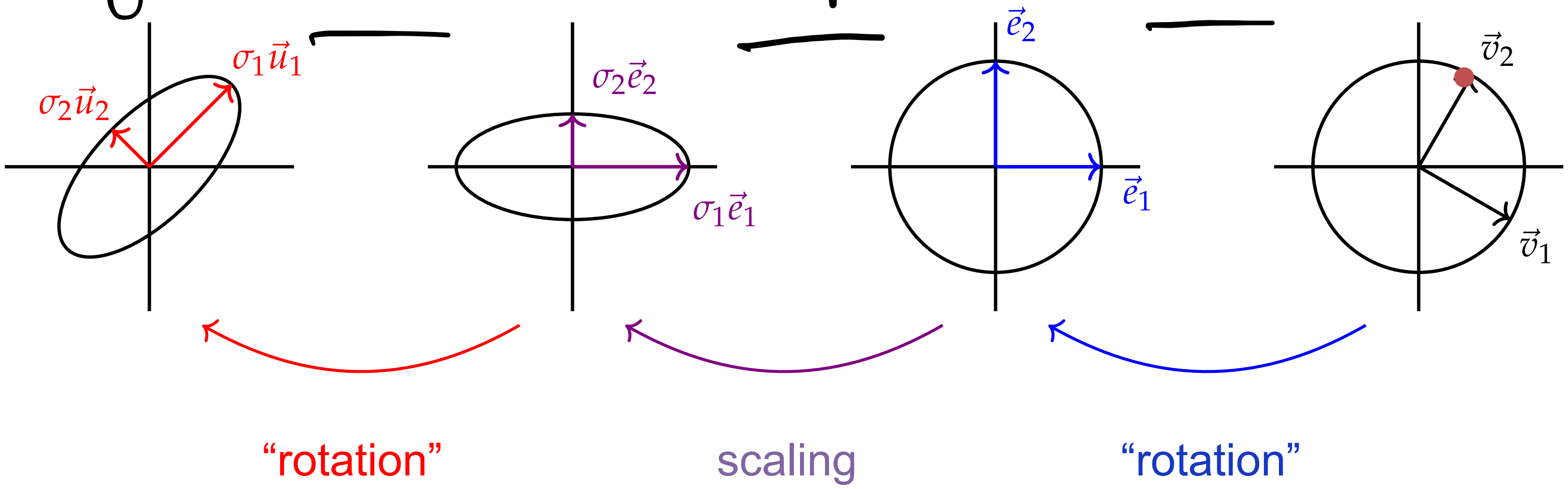
Cases:

1. **square and full rank (inverse);**
2. **full column rank (least squares, system identification);**
3. **full row rank (least norm, minimum energy control);**
4. **general cases: pseudo inverse, PCA etc.**

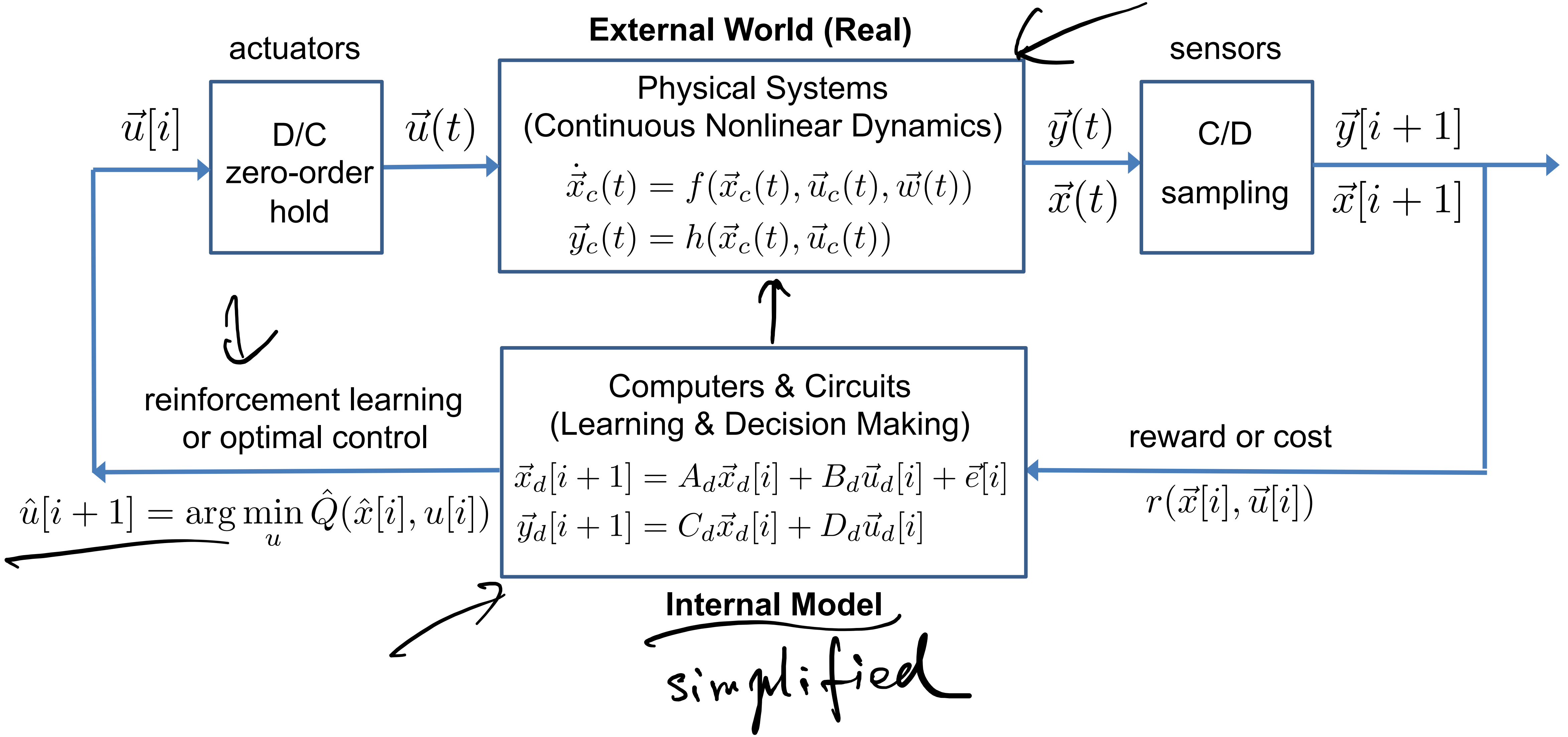
$$\vec{x}_\star = A^\dagger \vec{y}$$

$$A = y \leftarrow \underline{U}\ \underline{\Sigma_r}\ \underline{V^\top} \leftarrow x$$

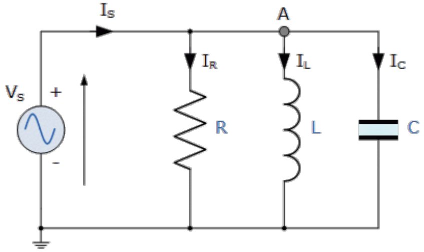
$$A^\dagger = V \Sigma_r^\dagger U^\top$$



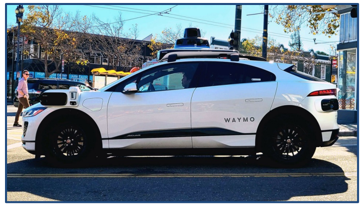
System Modeling, Analysis, & Control



System Modeling



?



mathematical modeling
from first principles

$$\begin{cases} \dot{\vec{x}}_c(t) = f(\vec{x}_c(t), \vec{u}_c(t), \vec{w}(t)) \\ \vec{y}_c(t) = h(\vec{x}_c(t), \vec{u}_c(t)) \end{cases}$$

approximation
& linearization

$$\begin{cases} \dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{n}(t) \\ \vec{y}(t) = C\vec{x}(t) + D\vec{u}(t) \end{cases}$$

discretization
& digitization

$$\begin{cases} \vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] + \vec{e}[i] \\ \vec{y}_d[i+1] = C_d\vec{x}_d[i] + D_d\vec{u}_d[i] \end{cases}$$

Discretization (Lecture 12)

$F = ma$

$$\rightarrow \quad \dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$$

$$\vec{x}(t) = e^{A(t-t_0)}\vec{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B\vec{u}(\tau)d\tau$$

$t_i = i\Delta$

$$\vec{x}_d[i+1] = \underbrace{e^{A\Delta}}_{A_d}\vec{x}_d[i] + \underbrace{\int_{i\Delta}^{(i+1)\Delta} e^{A(t-\tau)} Bd\tau}_{B_d}\,\vec{u}_d[i]$$

$$A_d = e^{A\Delta}$$

$$B_d = (e^{A\Delta} - I)A^{-1}B$$

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i]$$

$t = (i+1)\Delta$ $\quad$ $i\Delta$

System Modeling: Identification

Identification: **(Lecture 13)** $\vec{x}[i+1] = \underline{A}\vec{x}[i] + \underline{B}\vec{u}[i] + \vec{e}[i]$

From observations: $\vec{u}[0], \vec{u}[1], \ldots, \vec{u}[l], \ldots \leftarrow$

$\rightarrow \vec{x}[0], \vec{x}[1], \ldots, \vec{x}[l], \ldots \leftarrow$ $\quad \vec{y} = C\vec{x}$

$$\vec{y}\left\{\begin{array}{l} \vec{x}[1] = A\vec{x}[0] + B\vec{u}[0] \\ \vdots \\ \vec{x}[l+1] = A\underbrace{\vec{x}[l]}\,+\,\underbrace{B\vec{u}[l]} \end{array}\right.$$

$\underline{\vec{y}} = M\vec{x}$ $\quad \vec{x} = \begin{bmatrix} A \\ B \end{bmatrix}$

$\vec{x}[l] \quad \vec{u}[l]$

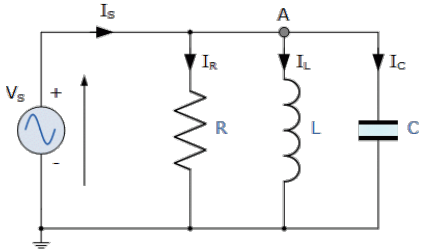
unknown (A, B) fixed

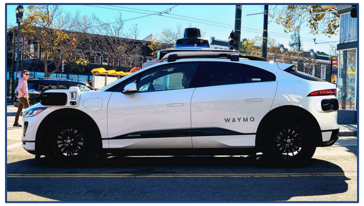
l very large

$\boxed{M}$ $\quad \min \|\vec{y} - M\vec{x}\|_2^2$

$\vec{x}_* \rightarrow [A_*, B_*]$

System Analysis





mathematical modeling
from first principles

$$\dot{\vec{x}}_c(t) = f(\vec{x}_c(t), \vec{u}_c(t), \vec{w}(t))$$
$$\vec{y}_c(t) = h(\vec{x}_c(t), \vec{u}_c(t))$$

approximation
& linearization

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{n}(t)$$
$$\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$$

discretization
& digitization

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] + \vec{e}[i]$$
$$\vec{y}_d[i+1] = C_d\vec{x}_d[i] + D_d\vec{u}_d[i]$$

Stability Criteria (Lecture 14)

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) \leftarrow$$

$$Re(\lambda_i(A)) < 0 \;, \; \forall i$$

$$\vec{x}_d[i+1] = \underline{A_d}\vec{x}_d[i] + B_d\vec{u}_d[i]$$

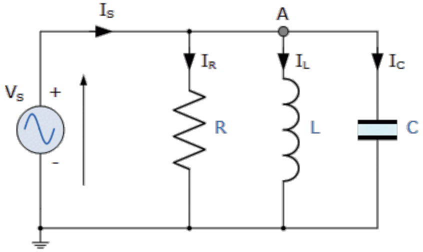
$$|\lambda_i(A_d)| < 1 \;, \; \forall i$$

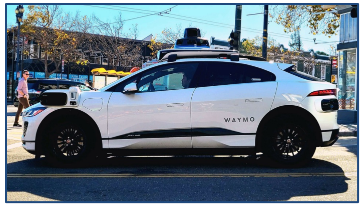
$$a \;, \; a_d = e^{a\Delta}$$

$$Re(a) < 0 \;, \; a_d = e^{Re(a) + \underline{j\,Im(a)}}$$

$$|a_d| = |e^{Re(a)}| \cdot \underline{|e^{j\,Im(a)}|}$$

System Control





mathematical modeling
from first principles

$$\dot{\vec{x}}_c(t) = f(\vec{x}_c(t), \vec{u}_c(t), \vec{w}(t))$$
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approximation
& linearization

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{n}(t)$$
$$\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$$

discretization
& digitization

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] + \vec{e}[i]$$
$$\vec{y}_d[i+1] = C_d\vec{x}_d[i] + D_d\vec{u}_d[i]$$

Controllability **(Lecture 15)**

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$$

$$\vec{u}[i] = F\vec{x}[i]$$

$$\vec{x}[i]$$

$$F$$

$$\vec{x}[i+1] = A\vec{x}[i] + BF\vec{x}[i]$$
$$= (A+BF)\vec{x}[i]$$

$\lambda_i(A_{cl})$ — stable?

System Control

Controllable Canonical Form: (Lecture 16)

$$\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \quad \in \mathbb{R}^n \qquad u = F\vec{x}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$F = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \end{bmatrix}$$

$$\det(\lambda I - A) =$$

$$\lambda^n - a_n\lambda^{n-1} - a_{n-1}\lambda a^{n-2} - \cdots - a_2\lambda - a_1$$

$$\det(\lambda I - A_{cl})$$

$$\lambda^n - (a_n + f_n)\lambda^{n-1} - \cdots - (a_2 + f_2)\lambda - (a_1 + f_1)$$

$$\underbrace{A + BF}_{A_{cl}} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & 0 & \\ a_1+f_1 & \cdots & & a_n+f_n \end{bmatrix}$$

$$(A, B)$$

System Control

Design control input to steer the state of a controllable system:

$\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ $\quad \mathcal{C} \doteq [A^{n-1}B \mid \cdots \mid AB \mid B] \in \mathbb{R}^{n\times n}$ is invertible.

$\mathcal{C}_\ell \doteq [A^{\ell-1}B \mid \cdots \mid AB \mid B] \in \mathbb{R}^{n\times \ell}$

$QC = I$ $\quad \vec{z} = T\vec{x}$ $\quad$ CCF

$Q = C^{-1}$ $\quad \vec{z}[i+1] = \underline{T^{-1}AT}\,\vec{z}[i] + T^{-1}Bu[i]$

$\vec{x}_f$? $\quad \vec{x}[0] \longrightarrow \vec{x}_f$ desired

$\vec{x}[1] = A\vec{x}[0] + Bu[0]$

$\vdots$

$\vec{x}[\ell+1] = A^\ell \vec{x}[0] + A^{\ell-1}Bu[0] + \cdots + Bu[\ell]$

$\vec{x}_f - A^\ell \vec{x}[0] = C_\ell\, \vec{u}[\ell]$

$\ell > n$

$\boxed{A = C_\ell}$

$\vec{y} = A\vec{x}$

$\min \|\vec{u}\|_2^2$

$\|\vec{u}\|_0$

System State Estimation

Estimate the state of the system from observable outputs:

$$\vec{x}_d[i+1] = A\vec{x}_d[i] + B\vec{u}_d[i] \leftarrow \vec{u}[i], \vec{x}[i]$$

$$\rightarrow \vec{y}_d[i+1] = C\vec{x}_d[i] + D\vec{u}_d[i]$$

$\boxed{}$

given $\vec{y} = C\vec{x}$ $\quad \vec{x}[0]$?

$\vec{y}[1] = C\vec{x}[1] = CA\vec{x}[0] + CB\vec{u}[0]$

$\vec{y}[2] = CA^2\vec{x}[0] + CAB\vec{u}[0] + CB\vec{u}[1]$

$\vdots$

$\vec{y} = A\vec{x}$

SVD, Low-Rank Approximation, PCA

$A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n] \in \mathbb{R}^{m \times n}$ $\quad A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top = U_r \Sigma_r V_r^\top$ $\quad \ell << r$

Low-rank Approximation: $\min_{B \in \mathbb{R}^{m \times n}} \|A - B\|_F^2$ subject to $\mathrm{rank}(B) = \ell$ **(Lecture 24)**

$$A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n] = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top = \underbrace{\sum_{i=1}^{\ell} \sigma_i \vec{u}_i \vec{v}_i^\top}_{B_*} + \sum_{i=\ell+1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

$$B_* = \sum_{i=1}^{\ell} \sigma_i \vec{u}_i \vec{v}_i^\top$$

SVD, Low-Rank Approximation, PCA

$A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n] \in \mathbb{R}^{m \times n}$ $\quad$ $\vec{\mu} = \frac{1}{n}(\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n) = \mathbf{0}$

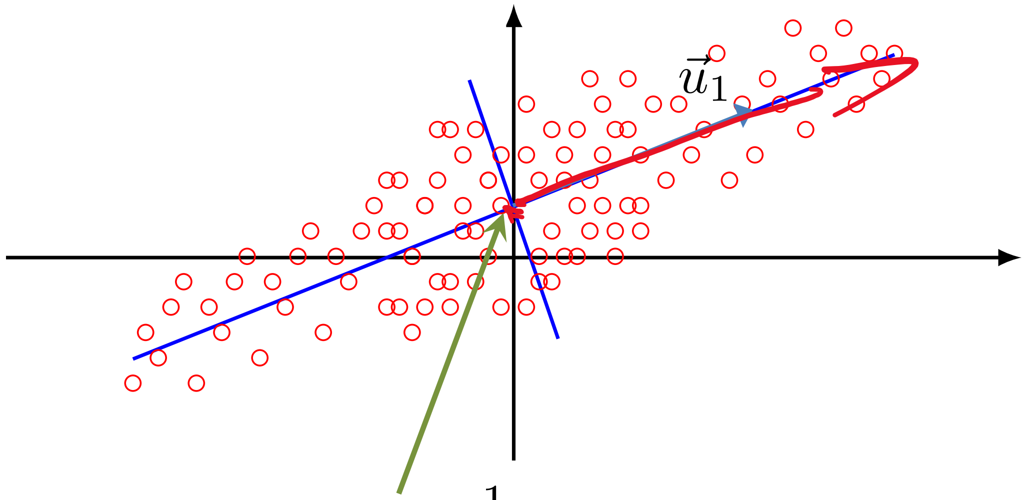
Principal Component: **(Lecture 24)**

find a normal vector $\|\vec{u}\|_2 = 1$ such that $\max_{\vec{u}} \|\vec{u}^\top A\|_2^2 = \|\vec{u}\vec{u}^\top A\|_2^2.$ $\Leftrightarrow \min \|A - B\|_2^2$

$\text{rank}(B) = 1$

$\vec{a}_i \Leftarrow \vec{a}_i - \vec{\mu}$ center

$B_* = \vec{u}_* \vec{u}_*^\top A = \sigma_1 \vec{u}_1 \vec{v}_1^\top$



$\vec{\mu} = \frac{1}{n}(\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n)$

SVD, Low-Rank Approximation, PCA

$$A = \underbrace{[\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]}\in \mathbb{R}^{m\times n} \qquad \vec{\mu} = \frac{1}{n}(\vec{a}_1 + \vec{a}_2 + \cdots + \vec{a}_n) = \mathbf{0}$$

Principal Components: **(Lecture 24)**

Find projection: $\max_{U_\ell} \|U_\ell U_\ell^\top A\|_F^2 \Leftrightarrow \min_{U_\ell} \|A - \underline{U_\ell U_\ell^\top} A\|_F^2 \Leftrightarrow \min_{U_{m-\ell}} \|U_{m-\ell} U_{m-\ell}^\top A\|_F^2$

$A \cdot B$

$\mathrm{rank}(B) \le \ell.$

$B = U_\ell U_\ell^\top$ proj