Note j: Complex Numbers

1 Complex Numbers Introduction

Definition 1 (Complex Numbers)
Consider an arbitrary complex number $a \in \mathbb{C}$. We can write this complex number as $a = x + \mathrm{j}y$ where $\mathrm{j} = \sqrt{-1}$ and $x, y \in \mathbb{R}$.

Definition 2 (Complex Number Operations)
Consider two complex numbers $a, b \in \mathbb{C}$. Let $a = x + \mathrm{j}y$ and $b = u + \mathrm{j}v$ where $x, y, u, v \in \mathbb{R}$. Addition is defined as follows:

$$a + b = (x + \mathrm{j}y) + (u + \mathrm{j}v) = (x + u) + \mathrm{j}(y + v) \tag{1}$$

and multiplication is defined as follows:

$$a \cdot b = (x + \mathrm{j}y) \cdot (u + \mathrm{j}v) = xu - yv + \mathrm{j}(xv + uy) \tag{2}$$

Note: this uses the "FOIL" technique for multiplication of real quantities.

Definition 3 (Complex Conjugate and Magnitudes)
Consider an arbitrary complex number $a \in \mathbb{C}$ where we can equivalently write $a = x + \mathrm{j}y$ for $x, y \in \mathbb{R}$. The complex conjugate of a is

$$\overline{a} = x - \mathrm{j}y \tag{3}$$

The magnitude of a is

$$|a| = \sqrt{a\overline{a}} \tag{4}$$

2 Polar Form

We will investigate another method to write complex numbers.

Theorem 4 (Euler's Identity)
Consider an arbitrary complex number $a \in \mathbb{C}$ which we can write as $a = x + \mathrm{j}y$. We can equivalently write this as $a = |a|e^{\mathrm{j}\theta}$ where $x = |a|\cos(\theta)$ and $y = |a|\sin(\theta)$ (equivalently, $\theta = \mathrm{atan2}(y, x)$[a]).

[a] Here, $\mathrm{atan2}(y, x)$ is a function that returns the angle from the positive x-axis to the vector from the origin to the point (x, y). See `https://en.wikipedia.org/wiki/Atan2`.

Proof. Let us write $a = |a|e^{\mathrm{j}\theta}$. We can show that $x = |a|\cos(\theta)$ and $y = |a|\sin(\theta)$, using the Taylor expansion of $f(x) = e^x$:

$$a = |a|e^{\mathrm{j}\theta} \tag{5}$$

$$= |a|\left(1 + \mathrm{j}\theta + \frac{(\mathrm{j}\theta)^2}{2!} + \frac{(\mathrm{j}\theta)^3}{3!} + \cdots + \frac{(j\theta)^{2n}}{2n!} + \frac{(\mathrm{j}\theta)^{2n+1}}{(2n+1)!} + \cdots\right) \tag{6}$$

$$= |a|\left(1 + \mathrm{j}\theta - \frac{\theta^2}{2!} - \mathrm{j}\frac{\theta^3}{3!} + \cdots + (-1)^n\frac{\theta^{2n}}{2n!} + \mathrm{j}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} + \cdots\right) \tag{7}$$

$$= |a|\left[\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + (-1)^n\frac{\theta^{2n}}{2n!} + \cdots\right) + \mathrm{j}\left(\theta - \frac{\theta^3}{3!} + \cdots + (-1)^n\frac{\theta^{2n+1}}{(2n+1)!} + \cdots\right)\right] \tag{8}$$

$$= |a|(\cos(\theta) + \mathrm{j}\sin(\theta)) \tag{9}$$

$$= \underbrace{|a|\cos(\theta)}_{x} + \mathrm{j}\underbrace{|a|\sin(\theta)}_{y} \tag{10}$$

To show that $\theta = \mathrm{atan2}(y, x)$, consider that

$$\frac{y}{x} = \frac{|a|\sin(\theta)}{|a|\cos(\theta)} \tag{11}$$

$$\implies \theta = \arctan\frac{y}{x} \tag{12}$$

Instead of using regular arctan, we will use atan2, two argument arctan, which protects against sign errors (i.e., to differentiate the cases when x and y are both positive or both negative) and division by zero (i.e., when $x = 0$). Hence, we write

$$\theta = \mathrm{atan2}(y, x) \tag{13}$$

□

The plot in Figure 1 visually describes the conversion from rectangular (i.e., $x + \mathrm{j}y$) form to polar form (i.e., $|a|\mathrm{e}^{\mathrm{j}\theta}$)

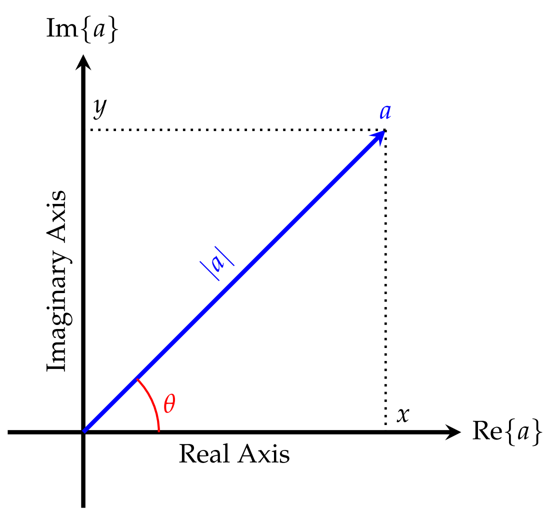

Figure 1: Complex number $a \in \mathbb{C}$ depicted as a vector in the complex plane.

Corollary 5 (Complex Exponential Representations of Sine and Cosine)
Using Theorem 4, we have that

$$\cos(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2} \tag{14}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{15}$$

Proof. Using Theorem 4 and the even/odd nature of cosine/sine respectively, we have the following direct results:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{16}$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta) \tag{17}$$

From this, we have that

$$2\cos(\theta) = e^{j\theta} - j\sin(\theta) + e^{-j\theta} + j\sin(\theta) \tag{18}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{19}$$

and

$$2j\sin(\theta) = e^{j\theta} - \cos(\theta) - e^{-j\theta} + \cos(\theta) \tag{20}$$

$$2j\sin(\theta) = e^{j\theta} - e^{-j\theta} \tag{21}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{22}$$

$\square$

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