EECS 16B Notes

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Introduction

These are notes for EECS 16B, a freshman-level survey of topics in electrical engineering. I (Simon) mostly wrote them during the Spring 2020 semester of EECS 16B paraphrasing Prof. Sanders's lectures the same semester. The sentences are my own; the source graphics[1] and a lot of proofreading are Seth's.

If you find a mistake or can suggest an improvement, let me know on GitHub.

The file you're viewing was compiled at 2020-10-01 20:59:50-07:00.

[1]which I lightly edited in GIMP

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Lecture 1

16A review and prerequisites

1.1 The language of circuits

Electrical circuits are models, specifically, abstractions of underlying physics-based descriptions of realities that govern behavior of an electrical system under analysis. Mathematically, circuits are collections of *nodes* joined by *branch elements*. Between every pair of adjacent nodes there is a *voltage difference*, measured in volts, as well as a *current*, measured in amps. You should be able to explain, both in approximate physical terms, and, if possible, by a mechanical analog, what voltage and current are. Given a circuit drawing, you should be able to write a comprehensive set of voltage-current constraints that fully predicts what is happening in the circuit. For a well-posed circuit model with N nodes, one preferred method is Nodal Analysis, which involves writing $N - 1$ linearly independent KCL node equations, and incorporating KVL and element branch consraints while writing the node equations.

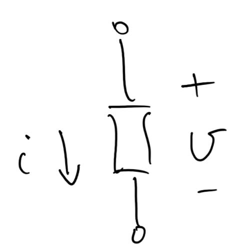

Figure 1.1: Current and voltage annotated on a passive element.

Understand how current and voltage are annotated on a circuit. Our terms are "voltage *across* branch element X" and "current *through* branch element X." The phrases "voltage *through*. . . " or "current *across*. . . " do not make sense. Understand, as shown in Fig. 1.1, that the reference directions for voltage

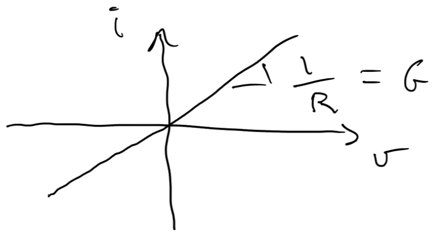

Figure 1.2: I-V characteristic of a resistor.

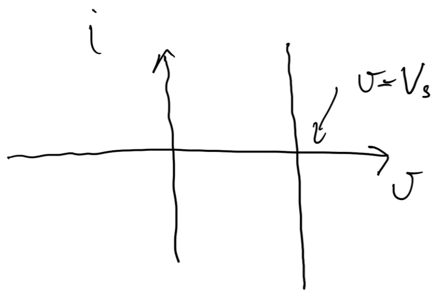

Figure 1.3: I-V characteristic of a voltage source.

and current are such that power absorbed by a circuit element is given by the formula vi.

1.2 Current-voltage characteristic

Resistor

As shown in Fig. 1.2, resistors enforce a proportionality relationship between current and voltage:

$$V = RI \tag{1.1}$$

$$I = GV \tag{1.2}$$

The ratio V/I is called *resistance*. The ratio I/V is called *conductance*.

Voltage source

As shown in Fig. 1.3, a voltage source will provide any current (or none at all) to maintain its target voltage.

Current source

As shown in Fig. 1.4, a current source will provide any voltage (or none at all) to maintain its target current.

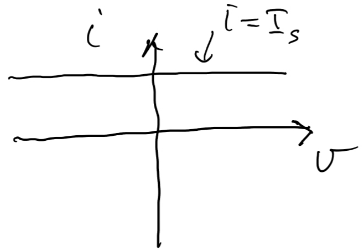

Figure 1.4: I-V characteristic of a current source.

Circuit-solving techniques

Be familiar with the following methods for solving circuits:

- Series elements, e.g. two resistors in series
- Parallel elements, e.g. two resistors in parallel.
- Voltage and current dividers
- Kirchoff's voltage and current laws
- Norton and Thévenin equivalent circuits
- Nodal analysis
- Power calculations

1.3 Linear algebra

Know what a vector is. Know what eigenvalues and eigenvectors are, and know how to solve for the eigenvalues and eigenvectors of a matrix, by solving for the null space of $A - \lambda I$, where λ is an indeterminate. Know why this technique works.

Lecture 2

Transistor Circuits

2.1 MOSFET behavior at a low level

Transistors are nonlinear circuit elements that are integral to building digital electronics. We'll focus on a class of transistor called MOSFET (*metal-oxide semiconductor field-effect transistor*), of which there are two types, NMOS and PMOS. For the most part, we will view MOSFETs from a digital perspective as voltage-controlled switches (more on that later), but we'll first have a look at the analog world under the hood.

The physical makeup of a MOSFET is shown in Figure 2.1. It is a device built on a silicon substrate with three terminals: *source* (S), *drain* (D), and *gate* (G). What makes a transistor a transistor is 2) mediated by gate voltage. (No current enters the gate of a MOSFET: $I_G = 0$.) 1) a current-voltage characteristic between drain and source, These quantities are labeled on Figure 2.2. Notice that voltages are understood with reference to their difference from V_S, so:

- D-S current-voltage characteristic is between I_D and $v_{DS} = v_D - v_S$,
- parameterized by $v_{GS} = v_G - v_S$.

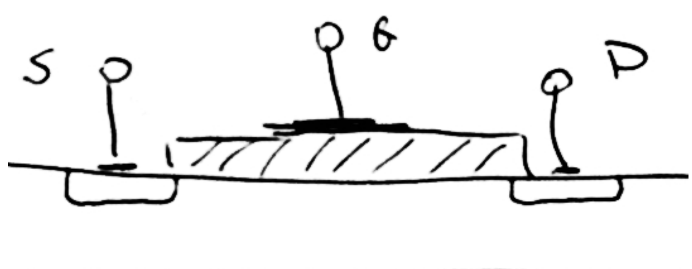

Figure 2.1: Physical construction of a simple MOSFET.

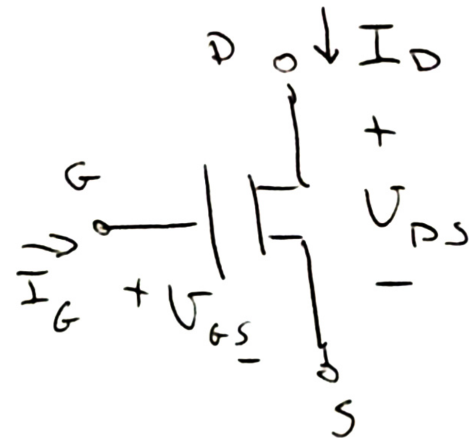

Figure 2.2: Currents and voltages labeled on an NMOS transistor.

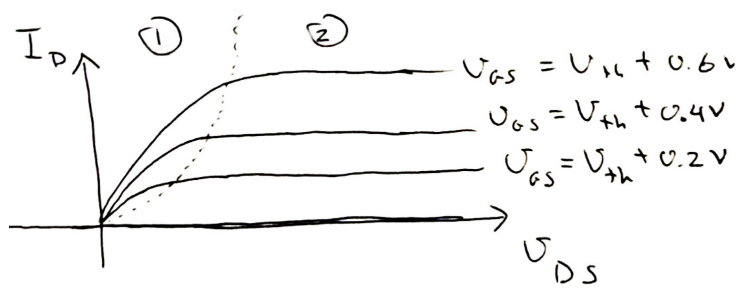

Figure 2.3: I-V characteristic of an NMOS transistor at different values of v_{GS}.

The role of v_{GS} in NMOS

Figure 2.3 depicts several current-voltage characteristics of an nmos, parameterized by v_{GS}. There's a lot happening on this graph in both the vertical and horizontal directions. Here's a self-guided tour:

- Notice the horizontal line lying along the positive v_{DS}-axis. This is the plot of the I-V characteristic when $v_{GS} < v_{t,n}$, where $v_{t,n} > 0$ is the threshold voltage for an NMOS transistor. The current-voltage characteristic is $I = 0$, the transistor is behaving as a current source corresponding to zero current—in other words, it's an open circuit. The transistor is "off." [1]
- Notice that three I-V curves, parameterized by how much v_{GS} exceeds $v_{t,n}$, lie above the line $I = 0$. Each of them is intersected by what looks like the eastern half of a dotted upward-facing parabola rising from the origin. This parabola divides the quadrant into two regions, one left and

[1] English semantics for "on" and "off" in circuits can be counterintuitive. An open circuit/switch is off, and vice versa.

region #	on/off?	v_{GS} predicate	v_{DS} predicate	name
0	off	$v_{GS} < v_{t,n}$	any	
1	on	$v_{GS} > v_{t,n}$	low	"linear region"
2	on	$v_{GS} > v_{t,n}$	high	"saturation"

Figure 2.4: Regions of an NMOS I-V characteristic.

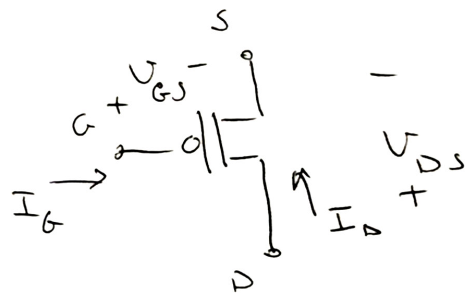

Figure 2.5: Currents and voltages labeled on a PMOS transistor.

one right. The left region is called region 1; the right region is called region 2.

- Focus on region 1, which is called the Linear Region. Notice that in region 1 near the origin, I_D and v_{DS} are proportional for every value of v_{GS}. The slope $G = I_D/v_{DS}$ increases for higher values of v_{GS}. This means that the D-S resistance $R = G^{-1}$ transitions from ∞ to a finite (perhaps small) value as v_{GS} increases past $v_{t,n}$. A resistor that can alternate between finite and infinite resistance is called a switch: in the Linear Region the transistor is a voltage-controlled switch.
- Focus on region 2, which is called Saturation, Here the I_D increases only very weakly as v_{DS} increases.For a given V_{DS}, I_D increases with increasing v_{GS}: the transistor behaves approximately as a voltage-controlled current source!

These characteristics are summarized in Figure 2.4. Regions 0 and 1 can be used to implement a switch. Region 2 is used for analog electronics—dependent sources, amplifiers, etc.

PMOS transistors: opposite of NMOS

Another kind of MOSFET is the PMOS. They have a similar construction as NMOS transistors, but their behavior is opposite, and for the "on" condition of $V_{GS} < V_{t,p}$, $V_{t,p} < 0$. Figure 2.5 and Figure 2.6 are the counterparts of Figure 2.2 and Figure 2.3, respectively.

For most of this class, we'll use more idealized models of these transistors in digital logic settings. In the voltage-controlled switch perspective, NMOS

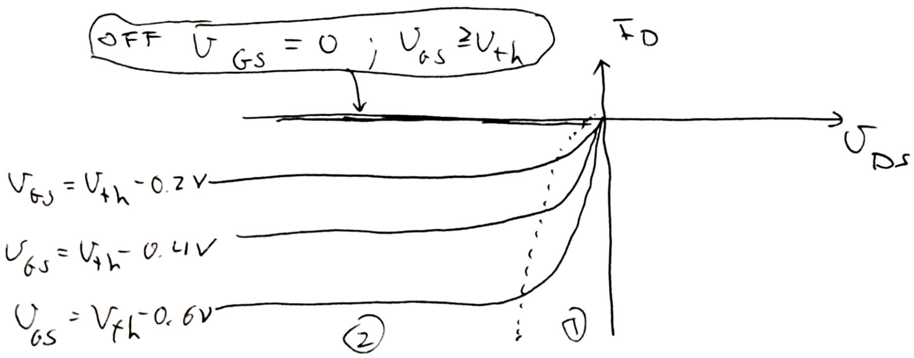

Figure 2.6: I-V characteristic of a PMOS transistor at different values of v_{GS}.

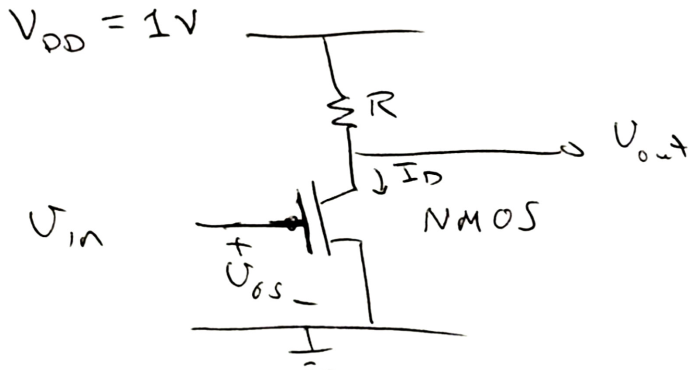

Figure 2.7: A inverter built using an NMOS transistor.

transistors open at lower voltages and close at higher voltages, and PMOS transistors close at lower voltages and open at high ones.

2.2 An NMOS inverter

One building block we need to understand digital logic is the *inverter*, which is a circuit that outputs a high voltage when its input is a low voltage, and vice versa. The high voltage represents a digital value of 1 (true), and the low voltage represents a digital value of 0 (false).

It's possible to build an inverter using an NMOS transistor, as shown in Figure 2.7. The high voltage is called V_{DD}, which stands for the voltage supplied by the high power rail, and in this example has a value of 1 volt.[2] In this example, our reference voltage will be ground—0 volts.

[2]For obscure historical reasons.

v_{in}	v_{out}
0	V_{DD}
V_{DD}	0

Figure 2.8: Truth table of the NMOS inverter.

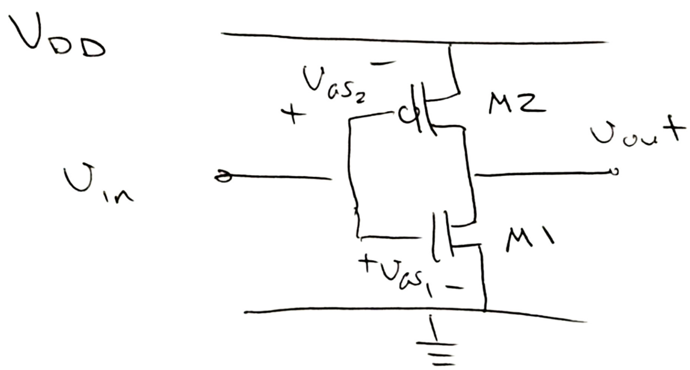

Figure 2.9: A inverter built using the CMOS design..

Analysis

- (Case $v_{in} = 0$) The transistor, as a switch, is off. As a result, the terminal v_{out} is connected directly to V_{DD} by a resistor. Because no current flows into the voltage terminal, by Ohm's law there can be no voltage drop across the resistor. Therefore $v_{out} = V_{DD}$.
- (Case $v_{in} = V_{DD}$) The transistor, as a switch, is on. The terminal v_{out} has a short to ground, so $v_{out} = 0$.

Figure 2.8 shows the truth table of this circuit and verifies that this circuit is indeed an inverter.

Power consumption

When $v_{in} = 0$, the circuit consumes no power, as we have established that there is no current through the resistor between V_{DD} and v_{out}. When $v_{in} = V_{DD}$, there is a path from V_{DD} through the resistor, then the transistor, to ground. The circuit consumes power $VI = V_{DD}^2/R$. While this might not necessarily be a lot, in computing applications with countless transistors, it adds up, and moving heat away from a dense circuit poses engineering challenges. Dense digital circuits were made possible by the discovery of the CMOS inverter architecture, which avoids a path from V_{DD} to ground.

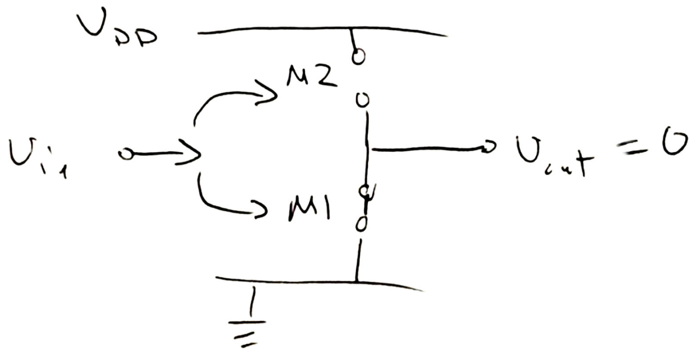

Figure 2.10: Equivalent circuit of Figure 2.9 when $v_{in} = V_{DD}$.

2.3 A CMOS inverter

Figure 2.9 shows an inverter circuit that exemplifies the *CMOS* design strategy of using PMOS and NMOS transistors together.

Analysis

- (Case $v_{in} = V_{DD}$)
  - The PMOS having as its source V_{DD} and v_{out} as its drain $V_{GS,1} = 0$, which is higher than $V_{t,p}$. Therefore there is no path from V_{DD} to v_{out}.
  - The NMOS having v_{out} as its drain and ground as its source has $V_{GS,2} = V_{DD}$, which is higher than $V_{t,n}$. Therefore, due to the terminal's short to ground, $v_{out} = 0$.

  The equivalent circuit once the switch model has been applied is shown in Figure 2.10.

- (Case $v_{in} = 0$)
  - The PMOS, having $V_{GS,1} = -V_{DD} < V_{t,p}$, turns on.
  - The NMOS, having $V_{GS,2} = 0 < V_{t,n}$, turns off.

  Therefore $V_{out} = V_{DD}$.

Power consumption

All currents are zero in this model, so no power is consumed.

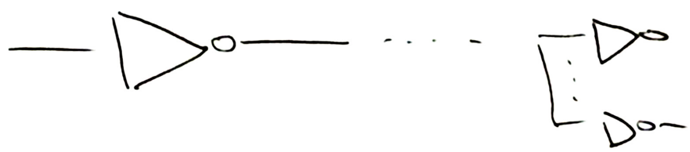

Figure 2.11: A chain of inverters, which is kind of similar to a computer.

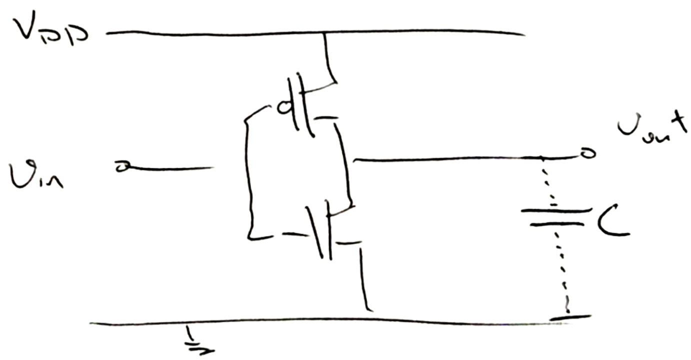

Figure 2.12: An inverter taken from a chain with a capacitor modeling the next stage.

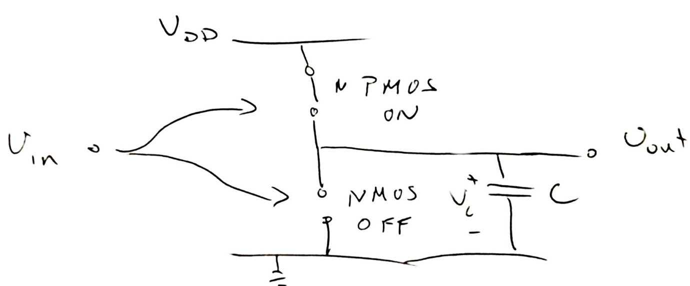

Figure 2.13: An inverter outputting V_{DD} with load capacitor.

2.4 A CMOS inverter chain with capacitance

Contrary to our last conclusion, inverters in real electronics certainly do consume some power. We'll pretend digital circuits are chains of inverters (Figure 2.11)—although this model won't teach you how to build a computer, it is close enough to real CMOS networks to illustrate when and where power is expended.

We will concentrate our analysis on just one stage of the CMOS inverter chain. A single inverter is shown in Figure 2.12, with a capacitor between v_{out} and ground to model the next stage's load. Figure 2.13 shows the equivalent circuit when the output of this inverter settles at V_{DD}.

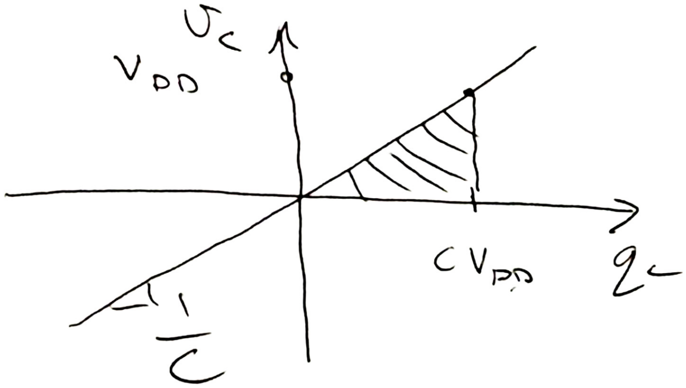

Figure 2.14: Energy stored in a capacitor can be computed by an integral under the $V = Q/C$ curve.

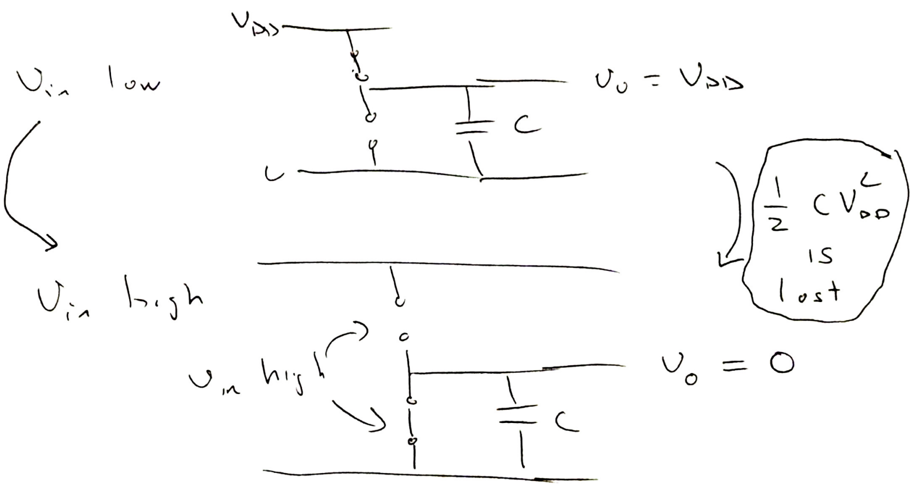

Figure 2.15: CMOS load capacitor forced from voltage V_{DD} to 0.

Potential energy in a capacitor

The energy stored in the capacitor when it has voltage V_{DD} is given by the formula

$$E_{cap} = \frac{1}{2}CV_{DD}^2, \tag{2.1}$$

which can be derived by using the facts that 1) that voltage is energy per unit charge and 2) a capacitor obeys $Q = CV$, and integrating through the total charge stored in the capacitor: $\int_0^{CV_{DD}} v_C \, dq$ (Figure 2.14).

When the inverter's input changes from low to high, the output must change from V_{DD} to 0 (Figure 2.15). That means that the load capacitor must discharge fully, burning $\frac{1}{2}CV_{DD}^2$ of potential energy as heat.

Total energy supplied

Even though the capacitor only stores and discharges $\frac{1}{2}CV_{DD}^2$, an up-down cycle costs CV_{DD}^2. This is because the voltage source must offer $Q = CV_{DD}$ of charge at V_{DD} energy per unit charge. Where does this go? Let's follow the energy as the output changes from 0 to 1 and back to 0.

1. $(q_C = 0, v_C = 0)$
2. Voltage source loads CV_{DD} of charge at V_{DD} energy per unit charge, at a total expense of CV_{DD}^2. Half of its energy output is burned by "parasitic" resistance en route to the capacitor, and the other half is stored in the capacitor.
3. $(q_C = CV_{DD}, v_C = V_{DD})$
4. Transistors toggle, and the capacitor drains, generating $\frac{1}{2}CV_{DD}^2$ of heat on the pull-down circuit.
5. $(q_C = 0, v_C = 0)$

Where does the energy in a device go?

With reference to our chain-of-inverters model, power consumption in digital devices is mainly explained by three phenomena:

- If the inverter flips every cycle at a clock speed of f_s, the circuit will burn $f_s CV_{DD}^2$ charging its capacitors.
- Leakage: a transistor that's "off" isn't 100% off, and a small amount of current flows and burns some energy.
- Short-circuit current (smaller): when the input is flipping between 0 and 1, there's a very short instant during which both transistors may be *on*, and some current flows through the momentary V_{DD}-ground short.

Lecture 3

Transient Analysis

(For this lecture, a MOSFET transistor is considered to transition between "on" and "off" at $v_{GS} = \frac{1}{2}V_{DD}$.)

We'll enrich our analog model of MOSFETS as voltage-controlled switches by acknowledging capacitance between the MOSFET's gate and source. Figure 3.1 and Figure 3.2 depict NMOS and PMOS transistors in this model.

Figure 3.3 summarizes the three levels of abstraction with which we are able to reason about CMOS inverters. On the very left is a digital symbol for an inverter that hides how the inverter works. In the center is the construction of an inverter using complementary MOSFETS. On the right is a fairly faithful analog representation of an inverter that will allow us to interrogate the assumptions that, thus far, have enabled us to treat the analog circuit as a digital one.

3.1 RC transient in an inverter chain

Let's return to the case study of a chain of inverters, this time focusing on just two consecutive inverters. In Figure 3.4 three wires are labeled as follows:

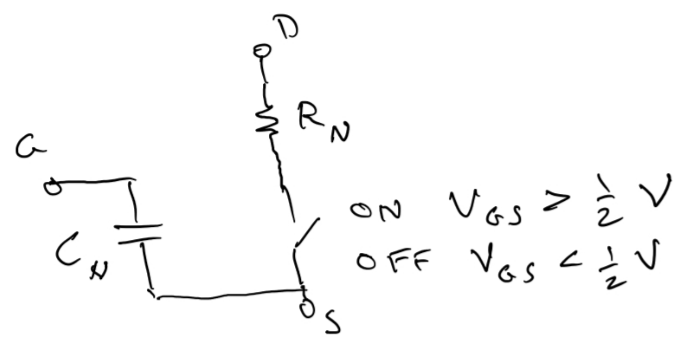

Figure 3.1: Model of NMOS transistor with G-S capacitance.

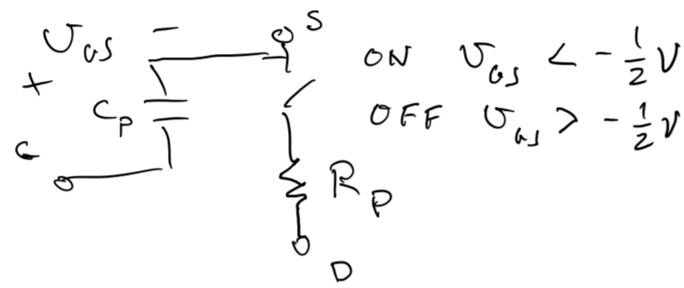

Figure 3.2: Model of PMOS transistor with G-S capacitance.

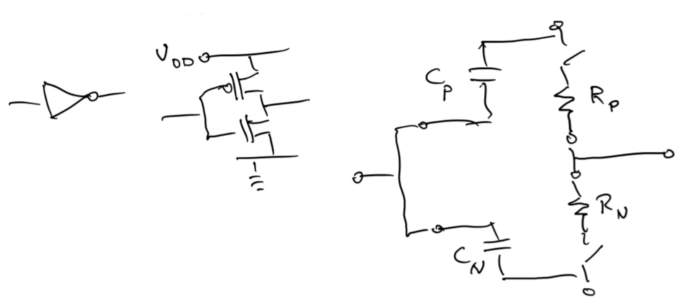

Figure 3.3: A CMOS inverter at three levels of abstraction.

Figure 3.4: Two consecutive CMOS inverters, part of a longer chain.

- v_{in} is the input to the first inverter,
- v_{o_1} is the output of the first inverter (and the input to the second), and
- v_{o_2} is the output of the second.

The digital logic interpretation is that v_{o_2} is the double negation of v_{in}, that is, $v_{o_2} = v_{\text{in}}$.

We will study what happens when v_{in} is driven by the input depicted in Figure 3.5. It will begin having remained at 0 for a long time, change to v_{DD} at time t_1, then return to 0 at time $t_2 > t_1$. Figure 3.6 shows the actions of the switches of the first inverter's transistors at times t_1 and t_2. For the rest of this section, we'll just concentrate on what happens to v_{o_1}.

Figure 3.5: Input signal to the first inverter of Figure 3.4

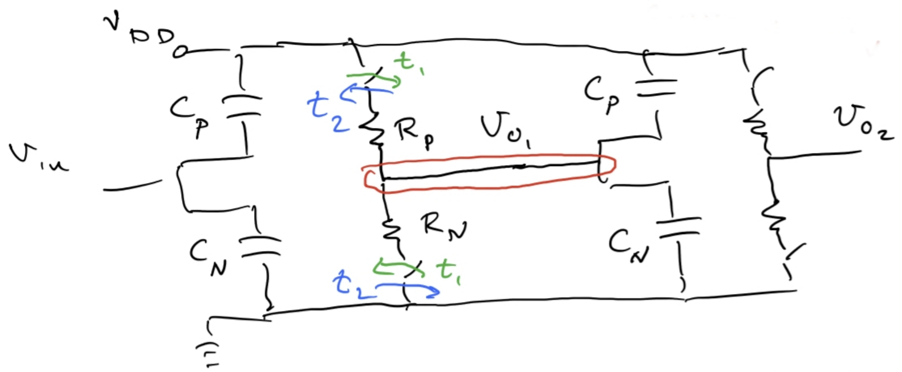

Figure 3.6: Analog redrawing of Figure 3.4, showing switch actions of the first inverter, as well as a distinguished node.

Before t_1

As $v_{in} = 0$ well before t_1, we can assume that the circuit has settled, and the output of the first inverter is v_{DD}.

After t_1, before t_2

At t_1, the pull-up switch opens, and the pull-down switch closes. KCL applied to the distinguished (red) middle node of Figure 3.6 requires the outgoing currents to sum to zero. Using Ohm's Law once and the capacitor current-voltage relationship twice, we have the following equation:

$$\frac{v_{o_1}}{R_N} + C_N \frac{d}{dt} v_{o_1} + C_P \frac{d}{dt} \left(v_{o_1} - v_{DD} \right) = 0 \tag{3.1}$$

$$\frac{d}{dt} v_{o_1} + \frac{1}{R_N \left(C_N + C_P \right)} v_{o_1} = 0 \tag{3.2}$$

This is a differential equation that we will analyze with initial condition $v_{o_1}(t_1) = V_{DD}$. For equations of this sort we will identify a characteristic quantity τ as follows:

$$\tau = R_N \left(C_N + C_P \right) \tag{3.3}$$

The International System of Units means that τ is measured in Ohm-Farads, or seconds. For this reason, τ is called the *time constant* of the system. A time constant on the order of tens of picoseconds is considered state-of-the-art for modern devices, arising from resistances on the order of kiloOhms and capacitances on the order of femtofarads. Rewriting using τ,

$$\frac{d}{dt} v_{o_1} = -\frac{1}{\tau} v_{o_1} \tag{3.4}$$

We will refer to the constant of proportionality between $\frac{d}{dt}v_{o_1}$ and v_{o_1} as λ.

$$\frac{d}{dt} v_{o_1} = \lambda v_{o_1} \tag{3.5}$$

There are many heuristic techniques to propose a solution to this differential equation. One of them is called Separation of Variables, which involves equations such as $\int \frac{dv_{o_1}}{v_{o_1}} = \int \lambda \, dt$. The resulting solution form, where A is a constant that remains to be determined, is all that you will need to know about this variety of differential equation:

$$v_{o_1}(t) = Ae^{\lambda t} \tag{3.6}$$

(As an aside, you can verify that $v_{o_1}(t) = Ae^{\lambda t}$ is a solution— differentiating both sides with respect to t results in $\frac{d}{dt}v_{o_1}(t) = A\lambda e^{\lambda t} = \lambda(Ae^{\lambda t})$.) Our next goal is to determine A. We can do so by choosing A to meet the initial condition $v_{o_1}(t_1) = V_{DD}$. Substituting $v_{o_1}(t) = Ae^{\lambda t}$,

$$Ae^{\lambda t_1} = V_{DD} \tag{3.7}$$

$$A = V_{DD} e^{-\lambda t_1} \tag{3.8}$$

$$v_{o_1} = \left(V_{DD} e^{-\lambda t_1}\right) e^{\lambda t} \tag{3.9}$$

$$= V_{DD} e^{-\left(\frac{t - t_1}{\tau}\right)} \tag{3.10}$$

Figure 3.7 is a sketch of v_{in} and v_{o_1} after t_1 and before t_2. Notice that v_{o_1} doesn't immediately jump to 0 like the digital model assumes. Rather, v_{o_1} decays exponentially toward 0 at a rate predicted by τ. Discharging a capacitor takes time, and digital devices' clock speed is limited by how quickly binary values settle in between logic gates.

After t_2

We will try to write a differential equation describing the evolution of v_{o_1} at time t_2 and beyond. Figure 3.6 shows that at time t_2, the pull-up switch closes, and the pull-down switch opens. KCL applied to the same central node yields

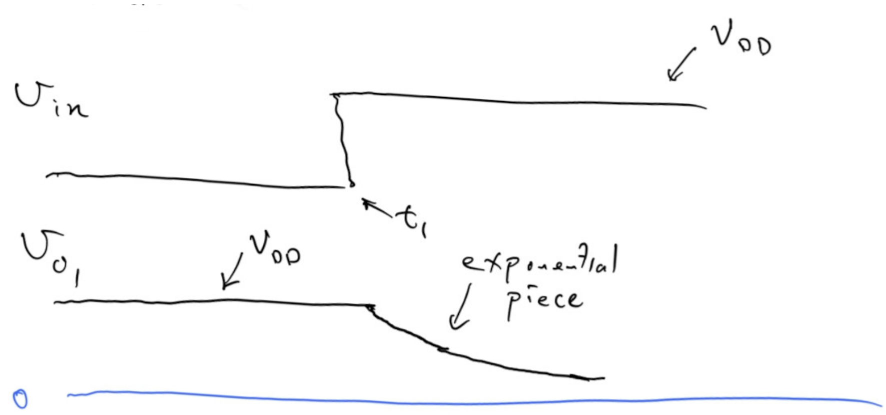

Figure 3.7: Sketch of transient from t_1 to t_2 in Figure 3.6.

the following differential equation:

$$\frac{v_{o_1} - V_{DD}}{R_P} + (C_P + C_N)\frac{d}{dt}v_{o_1} = 0 \tag{3.11}$$

$$\frac{d}{dt}v_{o_1} + \frac{1}{R_P(C_P + C_N)}v_{o_1} = \frac{v_{DD}}{R_P(C_P + C_N)} \tag{3.12}$$

The previous solution for v_{o_1}, which is valid up until time t_2, may be evaluated at t_2 for a boundary condition valid past t_2:

$$v_{o_1}(t_2) = V_{DD}e^{-\left(\frac{t_2 - t_1}{\tau}\right)} \tag{3.13}$$

A solution for v_{o_1} from t_2 onwards is:

$$v_{o_1} = V_{DD} + \left(v_{o_1}(t_2) - V_{DD}\right)e^{-\left(\frac{t - t_2}{\tau_P}\right)}, \tag{3.14}$$

where $\tau_P = R_P(C_P + C_N)$.

3.2 Uniqueness

We solved a differential equation. Differential equations are universal and ubiquitous in science and engineering.

A theorem states that a large class of differential equations with boundary conditions have unique solutions. These differential equations are of the form

$$\frac{d}{dt}x = f(x, t), \quad x(0) = x_0, \tag{3.15}$$

where

1. for all values of t, $f(x,t)$ is differentiable with respect to x and $\left|\frac{\partial f}{\partial x}(x,t)\right| < M$ for some nonnegative real number M; and

2. for all values of x, $f(x,t)$ has a finite number of discontinuities in t in any unit interval $[t_0, t_0 + 1]$.

If these conditions hold, then our differential equation has a unique solution.

Note that these conditions are in fact quite loose, and are more than enough to certify that unique solutions exist to differential equations of the form $\frac{d}{dt}x = f(x) = \lambda x$. It is important that we have proofs of existence and uniqueness because methods such as Separation of Variables are not inherently rigorous. Only once we have verifed that a proposed solution satisfies the differential equation and boundary condition may we claim that it is a solution. Because these problems have unique solutions, we may be certain that the model we are using is physically deterministic—it tells precisely what must happen, not just what *may* happen.

Lecture 4

Differential equations with inputs

4.1 RC with exponential input

In this section we will derive, in a more hands-on way, the behavior of an RC circuit forced by an exponential input. If you have ever used an ampwith knobs for treble and bass (Figure 4.1), then you have interacted with two circuits similar to the one shown in Figure 4.2. The resistor with a arrow is a variable resistor, or potentiometer,[1] that might be controlled by one of the amp's knobs.

In Figure 4.2,

- v_{in} represents the amp's analog input,
- v_o is used to drive the speakers after subsequent amplification, and
- R represents the setting on one of the potentiometers.

By studying the distinguished (green) node, we can write the following differential equation:

$$\frac{d}{dt} v_o(t) = -\frac{1}{RC} v_o(t) + \frac{1}{RC} v_{in}(t). \tag{4.1}$$

[1]Electric guitars use this circuit component, which guitarists call "pots," to blend the pickups' signals.

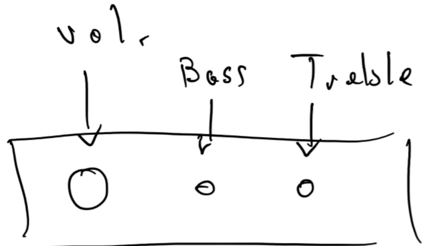

Figure 4.1: An amp with three knobs to adjust playback.

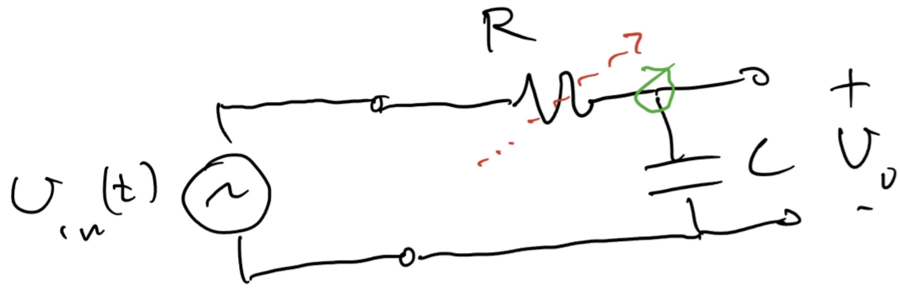

Figure 4.2: RC circuit as a filter.

We'll constrain v_{in} to have the following form:

$$v_{in}(t) = V_{in} e^{st}. \tag{4.2}$$

While it seems that this form is arbitrary, it will prove insightful, because e^{st} is an *eigenfunction* for input-output behavior of this circuit, i.e. we expect

$$v_o(t) = V_o e^{st}. \tag{4.3}$$

We can determine V_o by substituting our parameterization of v_o into Equation 4.1, whose LHS...

$$\frac{d}{dt} v_o(t) = \frac{d}{dt} V_o e^{st} \tag{4.4}$$

$$= sV_o e^{st} \tag{4.5}$$

... is equated with the RHS:

$$sV_o e^{st} = -\frac{1}{RC} V_o e^{st} + \frac{1}{RC} V_{in} e^{st}. \tag{4.6}$$

Now we can isolate V_o.

$$sV_0 + \frac{1}{RC} V_o = \frac{1}{RC} V_{in} \tag{4.7}$$

$$V_o = \left(\frac{1}{RC}\right)\left(\frac{1}{s + \frac{1}{RC}}\right) V_{in} \tag{4.8}$$

Substituting $\lambda = -\frac{1}{RC}$,

$$V_o = \frac{1}{1 - \frac{s}{\lambda}} V_{in} \tag{4.9}$$

All together, our solution for $v_o(t)$ is the following:

$$v_o(t) = V_o e^{st} = \frac{1}{1 - \frac{s}{\lambda}} V_{in} e^{st}. \tag{4.10}$$

Suppose that we have an initial condition for v_o at time 0.

$$v_o\Big|_{t=0} = v_1 \tag{4.11}$$

Then our solution, taking this fact into account, will be

$$v_o(t) = Ae^{-\frac{t}{RC}} + \frac{1}{RC}\left(\frac{V_{in}e^{st}}{s+\frac{1}{RC}}\right), \tag{4.12}$$

where A remains to be determined, viz. by evaluating both sides at $t = 0$:

$$v_1 = A + \frac{1}{RC}\left(\frac{V_{in}}{s+\frac{1}{RC}}\right) \tag{4.13}$$

$$A = v_1 - \frac{1}{RC}\left(\frac{V_{in}}{s+\frac{1}{RC}}\right). \tag{4.14}$$

This concludes our example. A solution to a linear differential equation will, generally, have the following structure:

$$v(t) = v_{\text{homogeneous}}(t) + v_{\text{particular}}(t), \tag{4.15}$$

where $v_{\text{homogeneous}}(t)$ corresponds to the initial condition, and $v_{\text{particular}}(t)$ to the input term.

$$\tag{4.16}$$

4.2 General scalar differential equation

We will verify that the following general differential equation:

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t); \quad x(t_0) = x_0 \tag{4.17}$$

has the following solution, which is a sum of a homogeneous and a particular term:

$$x(t) = e^{\lambda(t-t_0)}x_0 + \int_{t_0}^{t} e^{\lambda(t-\tau)}u(\tau)\,d\tau. \tag{4.18}$$

We can check the initial condition $x(t_0) = x_0$: the former term evaluates to x_0 and the latter to 0. Next, we can verify that $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$ holds by differentiating.

$$\frac{d}{dt}x(t) = \left\{\lambda e^{\lambda(t-t_0)}x_0\right\} + u(t) + \left\{\int_{t_0}^{t} \lambda e^{\lambda(t-\tau)}u(\tau)\,d\tau\right\} \tag{4.19}$$

The two terms in curly braces sum to $\lambda x(t)$, so Equation 4.17 is satisfied.

Lecture 5

Vector differential equations and second-order circuits

5.1 Guess-and-check for RC filter with cosine input

Last lecture we derived the folowing equation modeling the input-output properties of an amp: (where R is set by a potentiometer)

$$\frac{d}{dt}v_{out}(t) = -\frac{1}{RC}v_{out}(t) + \frac{1}{RC}v_{in}(t); \quad v_{out}\Big|_{t_0} = V \tag{5.1}$$

In this section we will try to determine the result in v_{out} when v_{in} has the following sinusoidal form:

$$v_{in}(t) = V_{in}\cos(\omega t) \tag{5.2}$$

This defines a sinusoid with amplitude V_{in} and a frequency of ω, which is angular frequency, in rad/s. Angular frequency is related to cycles/second by $\omega = 2\pi f$, where f is in units of Hz.

We can solve for v_{out} by guessing that the particular solution—-the summand that corresponds to v_{in}—has the form $A\cos(\omega t + \phi)$. The second summand of v_{out} is the homogeneous solution, which corresponds to the initial condition. It has the form $Be^{-\frac{1}{RC}(t-t_0)}$.

$$v_{out}(t) = A\cos(\omega t + \phi) + Be^{-\frac{1}{RC}(t-t_0)} \tag{5.3}$$

Substitution into the differential equation and initial conditions result in the following constants:

$$A = \frac{V_{in}}{\sqrt{\omega^2 (RC)^2 + 1}} \tag{5.4}$$

$$\phi = -\tan^{-1}(\omega RC) \tag{5.5}$$

$$B = v_{out}\Big|_{t_0} - A\cos(\omega t_0 + \phi) \tag{5.6}$$

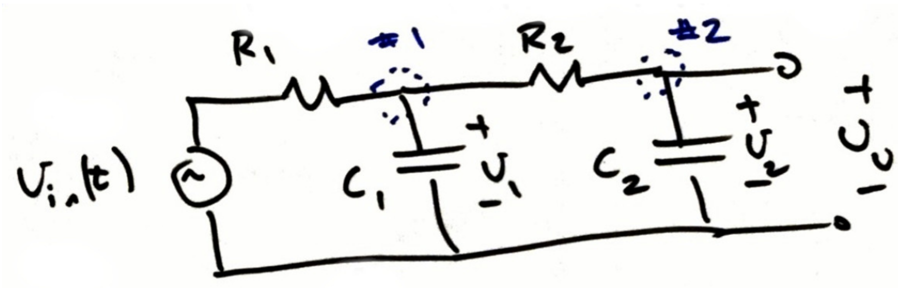

Figure 5.1: Filter with two resistors and two capacitors.

5.2 Second-order filter with two capacitors

Perhaps a "better" filter could be constructed by using two capacitors and two resistors instead of just one. Figure 5.1 depicts the proposed circuit, which is a "second-order circuit" or "second-order" filter, with values $C_1 = C_2 = 1\,\mu\text{F}$, $R_1 = \frac{1}{3}\,\text{M}\Omega$, and $R_2 = \frac{1}{2}\,\text{M}\Omega$. KCL at the two dotted-circled upper nodes yields:

$$C_1 \frac{d}{dt} v_1 + \frac{v_1 - v_{in}(t)}{R_1} + \frac{v_1 - v_2}{R_2} = 0 \tag{5.7}$$

$$C_2 \frac{d}{dt} v_2 + \frac{v_2 - v_1}{R_2} = 0 \tag{5.8}$$

In order to view this system of differential equations in state-space form, we will isolate derivatives on the LHS and emphasize that the RHS consists of linear combinations of v_1, v_2, and $v_{in}(t)$:

$$\frac{d}{dt} v_1 = -v_1 \left(\left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{1}{C_1} \right) + v_2 \left(\frac{1}{R_2 C_1} \right) + v_{in}(t) \left(\frac{1}{R_1 C_1} \right) \tag{5.9}$$

$$\frac{d}{dt} v_2 = \quad v_1 \left(\frac{1}{R_2 C_2} \right) - v_2 \left(\frac{1}{R_2 C_2} \right) \tag{5.10}$$

Written in matrix-vector form with physical parameters substituted,

$$\frac{d}{dt} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} v_{in}(t) \tag{5.11}$$

5.3 General state-space linear ODEs

Generally, a system of linear differential equations similar to the one derived above has the following form:

$$\frac{d}{dt} \vec{x} = A\vec{x} + \vec{b}u(t), \tag{5.12}$$

where $\vec{x}$ is a vector and A is a 2×2 matrix.

Suppose that A has an eigenvector $\vec{v}$ for an eigenvalue λ. We propose the following solution to the homogeneous problem $\frac{d}{dt}\vec{x} = A\vec{x}$:

$$\vec{x}(t) = \vec{v}e^{\lambda t} \tag{5.13}$$

and verify that "$\frac{d}{dt}\vec{x}$" and "$A\vec{x}$" for this candidate solution are equal:

$$\frac{d}{dt}\left(\vec{v}e^{\lambda t}\right) = \lambda\vec{v}e^{\lambda t} \tag{5.14}$$

$$A\left(\vec{v}e^{\lambda t}\right) = \lambda\vec{v}e^{\lambda t} \tag{5.15}$$

Detour: diagonalization of A

Let's additionally assume that A has two linearly independent eigenvectors:

$$A\vec{v}_1 = \lambda_1\vec{v}_1 \tag{5.16}$$

$$A\vec{v}_2 = \lambda_2\vec{v}_2 \tag{5.17}$$

These two relationships can be expressed simultaneously using matrices that consolidate the eigenvectors (side by side) and eigenvalues (on a diagonal):

$$A\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} \tag{5.18}$$

Calling the former two matrices V and the latter Λ,

$$AV = V\Lambda \tag{5.19}$$

Because we chose two linearly independent eigenvectors to constitute V, V is invertible. Stating A in terms of its eigenvectors and eigenvalues is called the *eigenvector-eigenvalue decomposition* of A:

$$A = V\Lambda V^{-1} \tag{5.20}$$

Second-order homogeneous solution from modes

Generally, $\vec{x}(0)$ will be a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{x}(0) = \tilde{x}_1(0)\vec{v}_1 + \tilde{x}_2(0)\vec{v}_2 \tag{5.21}$$

These coefficients can be solved by inverting V:

$$\begin{bmatrix}\tilde{x}_1(0) \\ \tilde{x}_2(0)\end{bmatrix} = V^{-1}\vec{x}(0) \tag{5.22}$$

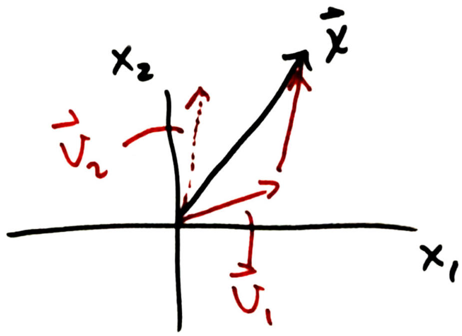

Figure 5.2: Decomposition of $\vec{x}$ along eigenbasis directions $\vec{v}_1$ and $\vec{v}_2$.

We can build a homogeneous solution for $\vec{x}(t)$ by superposing one-dimensional solutions in each eigenvector's respective direction:

$$\vec{x}(t) = \vec{v}_1 e^{\lambda_1 t}\tilde{x}_1(0) + \vec{v}_2 e^{\lambda_2 t}\tilde{x}_2(0) \tag{5.23}$$

$$= V \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{bmatrix} \tag{5.24}$$

To verify the initial condition, we can observe that the diagonal matrix of exponentials becomes an identity matrix at time 0:

$$\vec{x}(0) = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{bmatrix}, \tag{5.25}$$

which is true by construction (Equation 5.21).

Modal decomposition

In the previous section, we wrote $\vec{x}(0)$ in eigenbasis-aligned coordinates $\tilde{x}_1(0)$ and $\tilde{x}_2(0)$. In this section, we will follow $\tilde{x}_1$ and $\tilde{x}_2$ as functions of t. Recall that the eigenbasis-aligned coordinates are defined as follows:

$$\vec{x} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = V\vec{\tilde{x}}. \tag{5.26}$$

In reverse,

$$\vec{\tilde{x}} = V^{-1}\vec{x}. \tag{5.27}$$

We can use the Chain Rule to obtain a differential equation for $\vec{\tilde{x}}$:

$$\frac{d}{dt}\,\vec{\tilde{x}} = V^{-1}\,\frac{d}{dt}\,x \tag{5.28}$$

$$= V^{-1}\left(A\vec{x} + \vec{b}u\right) \tag{5.29}$$

$$= V^{-1}AV\vec{\tilde{x}} + V^{-1}\vec{b}u \tag{5.30}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\vec{\tilde{x}} + \vec{\tilde{b}}u, \quad \vec{\tilde{b}} = V^{-1}b \tag{5.31}$$

This vector differential equation is effectively scalar in each variable, in which scalar techniques can be applied separately. The separation of x into its eigenbasis-aligned components is called *modal decomposition*; $\vec{v}_1 e^{\lambda_1 t}$ and $\vec{v}_2 e^{\lambda_2 t}$ are the two *modes* of this system.

Lecture 6

Diagonalization to solve vector differential equations

In the last lecture, a second-order low-pass filter circuit using two resistors and two capacitors led us to the following differential equation:

$$\frac{d}{dt}\vec{x} = A\vec{x} + \vec{b}u, \tag{6.1}$$

where $\vec{x} = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ and $\vec{x}(0)$ or $\vec{x}(t_0)$ is known. We represented $\vec{x}$ as a linear combination of A's eigenvectors $\vec{v}_1$ (for eigenvalue λ_1) and $\vec{v}_2$ (for eigenvalue λ_2):

$$\vec{x} = \vec{v}_1\tilde{x}_1 + \vec{v}_2\tilde{x}_2 \tag{6.2}$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = V\vec{\tilde{x}} \tag{6.3}$$

We will assume that λ_1 and λ_2 are distinct, which implies that A has an invertible matrix of linearly independent eigenvectors V.We established that

$$AV = V\Lambda, \quad V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda 2 \end{bmatrix}. \tag{6.4}$$

These findings are summarized in Figure 6.1, which shows how $\vec{x}$, $A\vec{x}$, $\vec{\tilde{x}}$, and $\Lambda\vec{\tilde{x}}$ are related by matrix multiplication (along arrows).

6.1 Solution technique

A system

$$\frac{d}{dt}\vec{x} = A\vec{x} + \vec{b}u; \quad \vec{x}(t_0) \tag{6.5}$$

is solved as follows:

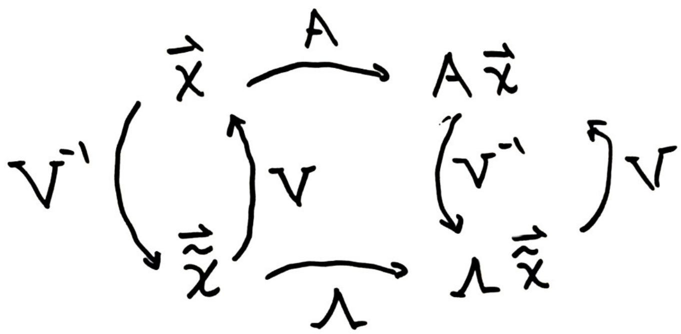

Figure 6.1: Illustration of multiplication actions of A, Λ, V, and V^{-1}.

1. Compute eigenvalues λ_1 and λ_2 of A, as well as their respective eigenvectors $\vec{v}_1$ and $\vec{v}_2$.
2. Construct $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ and define $\vec{\tilde{x}} = V^{-1}x$.
3. Construct $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $\tilde{b} = V^{-1}b$. Solve the differential equation $\frac{d}{dt}\vec{\tilde{x}} = \Lambda\vec{\tilde{x}} + \tilde{b}u$ with initial condition $\tilde{x}(t_0) = V^{-1}x(t_0)$. (More on this later.)
4. Recover a solution for $\vec{x}$ using $\vec{x} = V\vec{\tilde{x}}$.

6.2 Numerical example from RCRC circuit

Equation 5.11 captured a second-order low-pass filter using

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \tag{6.6}$$

We will solve the differential equation for $\vec{x} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ using the technique of the previous section.

Eigenvalues and eigenvectors

We will solve for eigenvectors λ as roots of $\det(\lambda I - A)$, the characteristic polynomial of A.

$$\det \begin{bmatrix} \lambda+5 & -2 \\ -2 & \lambda+2 \end{bmatrix} = \lambda^2 + 7\lambda + 6 = 0 \tag{6.7}$$

This quadratic equation in the indeterminate λ is called the *characteristic equation* of A. It has the following roots:

$$\lambda_1 = -1; \quad \lambda_2 = -6. \tag{6.8}$$

Next we will solve for an eigenvector belonging to eigenvalue λ_1, by choosing a nonzero vector from the null space of $\lambda_1 I - A$:

$$\lambda_1 I - A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \tag{6.9}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{6.10}$$

. . . and, *mutatis mutandis*, for λ_2:

$$\lambda_2 I - A = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \tag{6.11}$$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{6.12}$$

Differential equation in new coordinates

In our example,

$$V = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad \text{so} \tag{6.13}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}. \tag{6.14}$$

Our differential equation in $\vec{\tilde{x}}$ will be

$$\frac{d}{dt}\,\vec{\tilde{x}} = \Lambda\vec{\tilde{x}} + V^{-1}\vec{b}u \tag{6.15}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} \vec{\tilde{x}} + \begin{bmatrix} \frac{3}{5} \\ \frac{6}{5} \end{bmatrix}. \tag{6.16}$$

With $t_0 = 0$, $\vec{\tilde{x}}$ is solved as follows:

$$\vec{\tilde{x}}(t) = \big(\text{homogeneous solution}\big) + \big(\text{particular solution}\big) \tag{6.17}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \vec{\tilde{x}}(0) + \int_0^t \begin{bmatrix} e^{\lambda_1 (t-\tau)} & 0 \\ 0 & e^{\lambda_2 (t-\tau)} \end{bmatrix} \vec{\tilde{b}}u(\tau)\,d\tau, \tag{6.18}$$

viz., in individual components,

$$\begin{cases} \tilde{x}_1(t) = e^{\lambda_1 t}\tilde{x}_1(0) + \int_0^t e^{\lambda_1 (t-\tau)}\tilde{b}_1 u(\tau)\,d\tau \\ \tilde{x}_2(t) = e^{\lambda_2 t}\tilde{x}_2(0) + \int_0^t e^{\lambda_2 (t-\tau)}\tilde{b}_2 u(\tau)\,d\tau \end{cases} \tag{6.19}$$

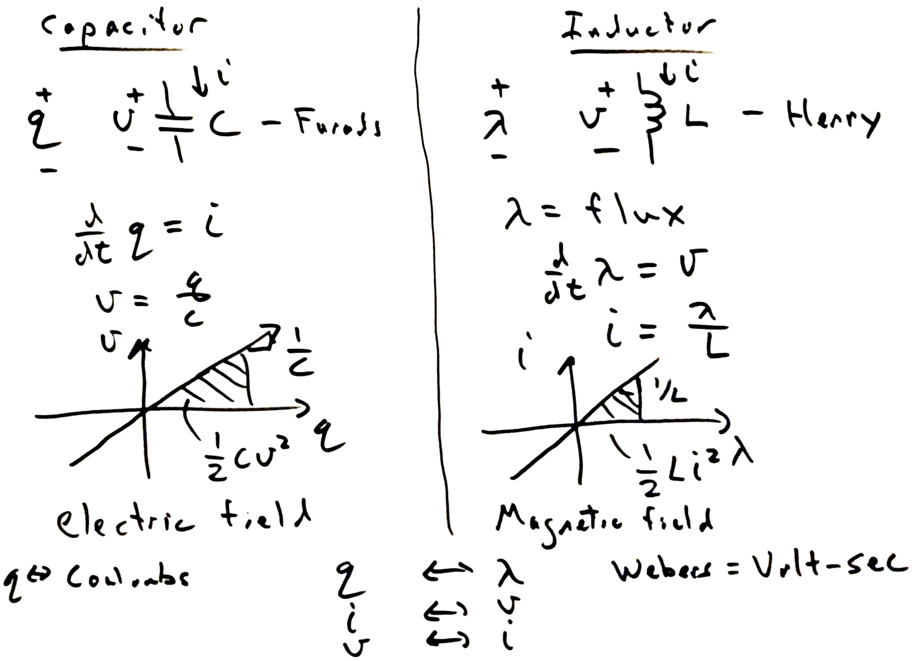

Figure 6.2: Parallels between capacitors and inductors.

Solution in original coordinates

A solution for $\vec{x}(t)$ may be reconstituted from eigenbasis-aligned coordinates using the following equation:

$$\vec{x}(t) = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}. \tag{6.20}$$

6.3 Introduction to inductors

Inductors are a branch element that are analogous to capacitors. Figure 6.2 compares them with capacitors, and the parallels are repeated below.

$$q = \text{charge (Coulomb)} \qquad \lambda = \text{flux (Weber = Volt-second)} \tag{6.21}$$

$$\frac{d}{dt} q = i \qquad \frac{d}{dt} \lambda = v \tag{6.22}$$

$$v = \frac{q}{C} \qquad i = \frac{\lambda}{L} \tag{6.23}$$

$$E_C = \frac{1}{2} C v^2 \qquad E_L = \frac{1}{2} L i^2 \tag{6.24}$$

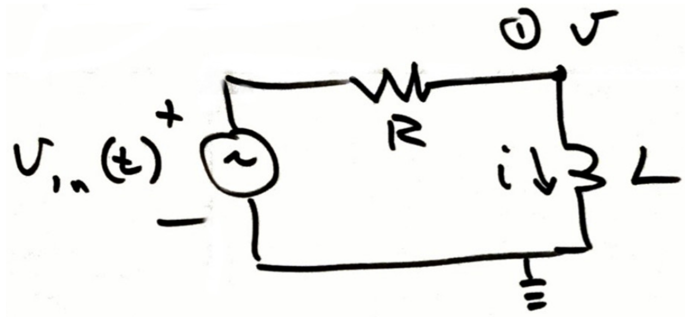

Figure 6.3: RL circuit, which is similar to an RC circuit (cf. Figure 4.2).

6.4 Example: RL circuit

Figure 6.3 shows a circuit with a time-varying voltage source, a resistor, and an inductor. KCL at the marked upper right node yields

$$\frac{v - v_{in}}{R} + i = 0. \tag{6.25}$$

In addition, from the current-voltage relationship of an inductor,

$$L \frac{d}{dt} i = v. \tag{6.26}$$

Eliminating v and isolating $\frac{d}{dt} i$, we have

$$\frac{d}{dt} i = -\frac{R}{L} i + \frac{v_{in}}{L}. \tag{6.27}$$

The state variable for an inductor is i, and this differential equation may be solved the same way we solved RC circuits.

Lecture 7

Inductors and RLC Circuits

7.1 LR

The RL circuit in Figure 7.1 can be described by the following differential equation:

$$\frac{d}{dt}i = -\frac{R}{L}i + \frac{1}{L}v_{in}. \tag{7.1}$$

It has eigenvalue $\lambda = -\frac{R}{L}$ and time constant $\tau = \frac{L}{R}$. Inductance is a ratio of magnetic flux (volt-seconds) to current (amps), so inductance divided by resistance works out to units of seconds.

7.2 LC

The LC circuit in Figure 7.2 can be described by the following two differential equations, which originate in Kirchoff's voltage and current laws, respectively.

$$C\frac{d}{dt}v + i = 0 \tag{7.2}$$

$$L\frac{d}{dt}i = v \tag{7.3}$$

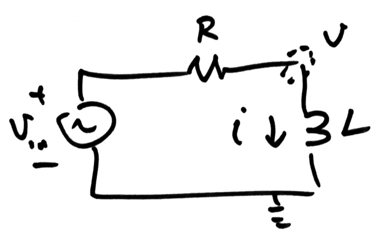

Figure 7.1: An RL circuit with an (AC) voltage source.

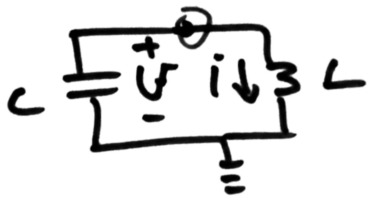

Figure 7.2: An LC circuit.

In vector form, with $\vec{x} = \begin{bmatrix} v \\ i \end{bmatrix}$,

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \vec{x} \tag{7.4}$$

This equation can be solved given an initial condition (knowing the value $\vec{x}(t_0)$ at some particular time t_0), yielding a solution for v and i that is good at every point in time. Substituting $L = 1\,\text{H}$ and $C = 1\,\text{F}$,

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x} \tag{7.5}$$

We will analyze this system by taking eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Its eigenvalues are the roots of its characteristic polynomial $\det(\lambda I - A) = \lambda^2 + 1$, which are $\pm j$. Call them $\lambda_1 = j$ and $\lambda_2 = -j$. Solving for eigenvectors,

$$\lambda_1 \rightsquigarrow \begin{bmatrix} j & 1 \\ -1 & j \end{bmatrix} \vec{v}_1 = \vec{0} \implies \vec{v}_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix} \tag{7.6}$$

$$\lambda_2 \rightsquigarrow \begin{bmatrix} -j & 1 \\ -1 & -j \end{bmatrix} \vec{v}_2 = \vec{0} \implies \vec{v}_2 = \begin{bmatrix} 1 \\ j \end{bmatrix} \tag{7.7}$$

Conjugate pairs

Notice that $\lambda_2 = \overline{\lambda_1}$ and $\vec{v}_2 = \overline{\vec{v}_1}$. The eigenvalues and eigenvectors come in conjugate pairs because A is real, and the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$ has real-valued coefficients. The Fundamental Theorem of Algebra states that a polynomial of degree n has n roots. When the polynomial has real coefficients, then roots are real or occur in complex conjugate pairs: let $p(\lambda) = 0$. Then $\overline{p(\lambda)} = \overline{0} = 0$:

$$\overline{p(\lambda)} = \overline{\lambda^n} + \overline{a_{n-1}\lambda^{n-1}} + \ldots + a_1\overline{\lambda} + a_0 = 0 \tag{7.8}$$

$$= \left(\overline{\lambda}\right)^n + a_{n-1}\left(\overline{\lambda}\right)^{n-1} + \ldots + a_1\overline{\lambda} + a_0 = 0 \tag{7.9}$$

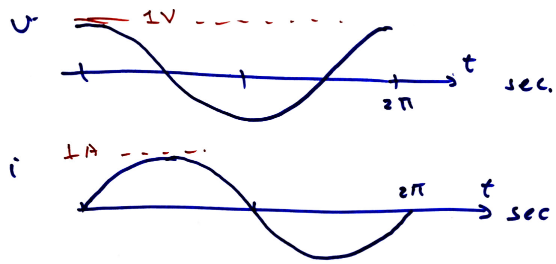

Figure 7.3: Current and voltage of an oscillating LC circuit.

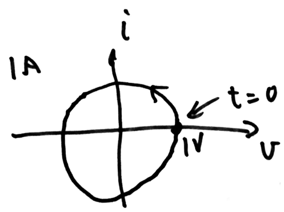

Figure 7.4: Phase portrait of an oscillating LC circuit.

From this we can see that whenver λ_1 is a root of the real polynomial $p(\lambda)$, so is $\overline{\lambda}_1$.

Conjugate pairing also happens with eigenvectors when A is real. If $A\vec{v}_1 = \lambda_1\vec{v}_1$, then conjugating both sides, we have

$$\overline{(A\vec{v}_1)} = \overline{(\lambda_1\vec{v}_1)} \tag{7.10}$$

$$A\overline{\vec{v}_1} = \overline{\lambda}_1\overline{\vec{v}_1} \tag{7.11}$$

This shows that $\overline{\vec{v}_1}$ is an eigenvector as well, completing a pair with $\vec{v}_1$.

Back to LC circuit

Let's take initial condition $\vec{x}(0) = \begin{bmatrix} 1\,\text{V} \\ 0\,\text{A} \end{bmatrix}$. As a combination of $\vec{v}_1$ and $\vec{v}_2$,

$$\vec{x}(0) = \frac{1}{2}\underbrace{\begin{bmatrix} 1 \\ -j \end{bmatrix}}_{\vec{v}_1} + \frac{1}{2}\underbrace{\begin{bmatrix} 1 \\ j \end{bmatrix}}_{\vec{v}_2} \tag{7.12}$$

Therefore, the same linear combination will construct $\vec{x}(t)$ from its constituent modes.

$$\vec{x}(t) = \frac{1}{2}\underbrace{\begin{bmatrix}1\\-j\end{bmatrix}}_{\vec{v}_1} e^{jt} + \frac{1}{2}\underbrace{\begin{bmatrix}1\\j\end{bmatrix}}_{\vec{v}_2} e^{-jt} \tag{7.13}$$

$$= \begin{bmatrix}\frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}\\ \frac{1}{j}e^{jt} - \frac{1}{2j}e^{-jt}\end{bmatrix} = \begin{bmatrix}\cos t\\ \sin t\end{bmatrix}, \tag{7.14}$$

which follows from the identities $\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$ and $\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$, which are both consequences of Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$.

Under an oscilloscope, $v(t) = \sin t$ and $i(t) = \cos t$ might appear as they do in Figure 7.3. If $\vec{x}(t)$ is plotted as a parametric curve in the plane (called a *phase portrait*), the result is a counterclockwise traversal of the unit circle, shown in Figure 7.4. This means that the $\vec{x}$ vector has constant length:

$$v^2 + i^2 = 1 \text{ for all } t. \tag{7.15}$$

Euler's formula

Euler's formula states that $e^{j\theta} = \cos\theta + j\sin\theta$. This can be derived from the series expansion of the exponential function around $\theta = 0$,

$$e^z = 1 + z + \frac{1}{2!}z^2 + \dots \tag{7.16}$$

Substituting $z = j\theta$, we have

$$e^{j\theta} = 1 + j\theta + \frac{-1}{2!}\theta^2 + \frac{-j}{3!}\theta^3 + \dots, \tag{7.17}$$

whose even terms add to $\cos\theta$:

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots, \tag{7.18}$$

and whose odd terms add to $j\sin\theta$:

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 \dots. \tag{7.19}$$

7.3 LRC

The following node equations describe the LRC circuit in Figure 7.5:

$$(1) \qquad -i + \frac{v_1 - v}{R} = 0 \tag{7.20}$$

$$(2) \qquad \frac{v - v_1}{R} + C\frac{d}{dt}v = 0 \tag{7.21}$$

$$(3) \qquad L\frac{d}{dt}i = -v_1 \tag{7.22}$$

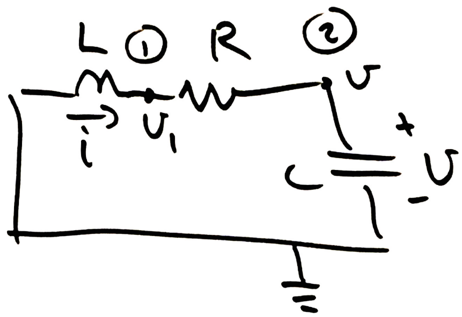

Figure 7.5: An LRC circuit.

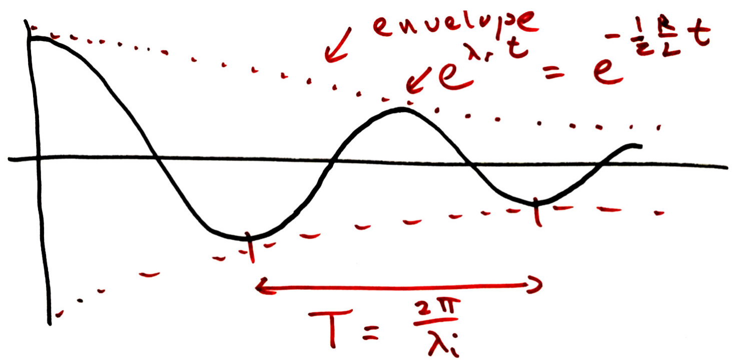

Figure 7.6: The effects of the real and imaginary parts of an eigenvalue $\lambda = \lambda_r + j\lambda_i$, when neither is zero.

After using Equation 7.20 to eliminate v_1, we have the following two differential equations:

$$C \frac{d}{dt} v = i \tag{7.23}$$

$$L \frac{d}{dt} i = -Ri - v \tag{7.24}$$

or, in vector form,

$$\frac{d}{dt} \begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} \tag{7.25}$$

From its characteristic equation

$$0 = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}, \tag{7.26}$$

we obtain eigenvalues

$$\lambda_{1,2} = -\frac{1}{2}\frac{R}{L} \pm j\sqrt{\left(\frac{1}{2}\frac{R}{L}\right)^2 - \frac{1}{LC}}. \tag{7.27}$$

As R increases from zero, the eigenvalues move in significant ways.

1. When $R = 0$, the eigenvalues are the imaginary pair $\pm j\sqrt{\frac{1}{LC}}$.
2. When $\left(\frac{1}{2}\frac{R}{L}\right)^2 = -\frac{1}{LC}$, the eigenvalues are both $-\frac{1}{2}\frac{R}{L}$.
3. When $\left(\frac{1}{2}\frac{R}{L}\right)^2 > -\frac{1}{LC}$, the eigenvalues are distinct and real-valued.

Between (1) and (2), the eigenvalues are neither real nor pure imaginary:

$$\lambda_{1,2} = \underbrace{-\frac{1}{2}\frac{R}{L}}_{\lambda_r} \pm j\underbrace{\sqrt{\left(\frac{1}{2}\frac{R}{L}\right)^2 - \frac{1}{LC}}}_{\lambda_i} = \lambda_r \pm j\lambda_i \tag{7.28}$$

The time domain response will incorporate $e^{\lambda t}$, which has the following trigonometric interpretation:

$$e^{(\lambda_r + j\lambda_i)t} = e^{\lambda_r t}e^{j\lambda_i t} \tag{7.29}$$

$$= e^{\lambda_r t}\left(\cos\lambda_i t + j\sin\lambda_i t\right), \tag{7.30}$$

a sinusoid of angular frequency λ_i under an envelope with rate λ_r, as shown in Figure 7.6.

Lecture 8

Phasors

8.1 Exponential inputs and outputs

Phasors are a ubiquitous method for understanding particular responses of linear differential equations, given sinusoidal input. The following differential equation is familiar as the capacitor voltage of an RC circuit with v_{in} across both components:

$$\frac{d}{dt} v_o = -\frac{1}{RC} v_o + \frac{1}{RC} v_{\text{in}}, \quad \lambda = -\frac{1}{RC} \tag{8.1}$$

Equation 5.3 showed the solution to this differential equation with $v_{\text{in}} = V_{\text{in}} \cos \omega t$, purportedly obtained by direct substitution. While that works, an easier way to the solution is to practice on a general exponential input:

$$v_{\text{in}} = V_{\text{in}} e^{st}. \tag{8.2}$$

Assuming that $v_o = V_o e^{st}$ is also exponential with the same rate s,

$$sV_o e^{st} = -\frac{1}{RC} V_o e^{st} + \frac{1}{RC} V_{\text{in}} e^{st} \tag{8.3}$$

$$V_o = \frac{1}{1 + sRC} V_{\text{in}} = \frac{1}{1 - \frac{s}{\lambda}} V_{\text{in}} \tag{8.4}$$

If our circuit has not existed with its input forever, then the solution for v_o is a superposition of $V_o e^{st}$ with a homogeneous response, in which A is a constant that depends on the initial condition.

$$v_o = V_o e^{st} \underbrace{+ A e^{\lambda t}}_{\to 0 \ \text{if}\, \text{Re}\{\lambda\} < 0} \tag{8.5}$$

If λ has a negative real part, then the impact of the initial condition tends to zero as $t \to \infty$. Sinusoidal inputs and outputs arise as superpositions of exponentials where s has no real part:

$$s = j\omega \rightsquigarrow e^{st} = \cos\omega t + j\sin\omega t \tag{8.6}$$

$$\cos\omega t = \mathrm{Re}\left\{e^{j\omega t}\right\} = \mathrm{Re}\{\cos\omega t + j\sin\omega t\} \tag{8.7}$$

$$\cos\omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \tag{8.8}$$

$$\sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \tag{8.9}$$

8.2 Phasor representation of a sinusoid

A phasor is a complex number that has amplitude and phase information of a time-domain sinusoid. By writing the following cosine as the real part of a complex exponential, we can factor the phasor part from the time-varying part.

$$x(t) = A\cos(\omega t + \phi), \quad A \text{ real, positive} \tag{8.10}$$

$$= \mathrm{Re}\left[Ae^{j(\omega t+\phi)}\right] \tag{8.11}$$

$$= \mathrm{Re}\left[\underbrace{Ae^{j\phi}}_{\text{phasor}} e^{j\omega t}\right] \tag{8.12}$$

The phasor representation of $A\cos(\omega t + \phi)$ is $Ae^{j\phi}$. ϕ is the *phase* of this sinusoid.

Uniqueness

Phasors uniquely represent the sinusoids that they represent. Suppose we assign the phasors A_1 and A_2 to sinusoids x_1 and x_2, respectively:

$$x_1(t) = \mathrm{Re}\left\{A_1 e^{j\omega t}\right\} \xrightarrow{\text{phasor representation}} A_1 \tag{8.13}$$

$$x_2(t) = \mathrm{Re}\left\{A_2 e^{j\omega t}\right\} \xrightarrow{\text{phasor representation}} A_2 \tag{8.14}$$

Uniqueness means that $(x_1 = x_2) \iff (A_1 = A_2)$. We can see see that $A_1 = A_2$ implies $x_1 = x_2$, because the identity $\mathrm{Re}\left\{A_1 e^{j\omega t}\right\} = \mathrm{Re}\left\{A_2 e^{j\omega t}\right\}$ follows immediately from $A_1 = A_2$.

To show that $x_1 = x_2$ implies $A_1 = A_2$, we verify the real and imaginary layers of this equation independently. The real part emerges at $t = 0$: $x_1(0) = \mathrm{Re}\{A_1\}$ and $x_2(0) = \mathrm{Re}\{A_2\}$. Therefore $\mathrm{Re}\{A_1\} = \mathrm{Re}\{A_2\}$. On the other hand, we need $t = \frac{\pi}{2}\frac{1}{\omega}$ to access the imaginary part. From

$$x_1\Big|_{t=\frac{\pi}{2}\frac{1}{\omega}} = x_2\Big|_{t=\frac{\pi}{2}\frac{1}{\omega}} \tag{8.15}$$

follows

$$\text{Re}\left\{A_1 e^{j\frac{\pi}{2}}\right\} = \text{Re}\left\{A_2 e^{j\frac{\pi}{2}}\right\}. \tag{8.16}$$

Applying $e^{j\frac{\pi}{2}} = j$,

$$\text{Re}\,(A_1 j) = \text{Re}\,(A_2 j) \tag{8.17}$$

$$\text{Re}\left[\text{Re}\,(A_1)\, j + j^2\,\text{Im}\,(A_1)\right] = \text{Re}\left[\text{Re}\,(A_1)\, j + j^2\,\text{Im}\,(A_1)\right] \tag{8.18}$$

$$-\text{Im}\,(A_1) = -\text{Im}\,(A_2) \tag{8.19}$$

Linearity

Linearity of the phasor transformation means that a real linear combination of sinusoids is represented as the same linear combination of the sinusoids' respective phasors. For real constants a_1 and a_2, the phasor representation of $a_1 x_1(t) + a_2 x_2(t)$ is $a_1 A_1 + a_2 A_2$. Beginning with our two sinusoids as real parts of scaled complex exponentials,

$$x_1(t) = \text{Re}\left(A_1 e^{j\omega t}\right) \tag{8.20}$$

$$x_2(t) = \text{Re}\left(A_2 e^{j\omega t}\right), \tag{8.21}$$

we may form the following linear combination with real coefficients a_1 and a_2:

$$a_1 x_1(t) + a_2 x_2(t) = \text{Re}\left(a_1 A_1 e^{j\omega t}\right) + \text{Re}\left(a_2 A_2 e^{j\omega t}\right) \tag{8.22}$$

$$= \text{Re}\left[(a_1 A_1 + a_2 A_2)\, e^{j\omega t}\right]. \tag{8.23}$$

Differentiation

Phasors represent differentiation in time as multiplication by $j\omega$: if

$$x(t) = \text{Re}\left[A e^{j\omega t}\right], \tag{8.24}$$

then

$$\frac{d}{dt}\, x(t) = \text{Re}\left[\underbrace{j\omega A}_{\text{new phasor}}\; e^{j\omega t}\right]. \tag{8.25}$$

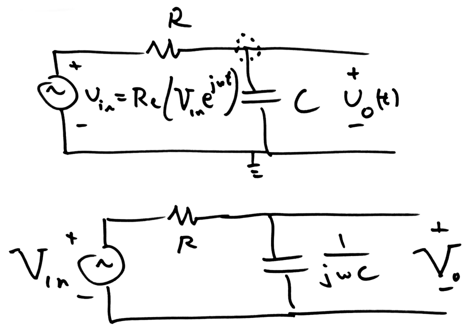

Figure 8.1: An RC circuit with a sinusoidal voltage source and its phasor domain representation.

8.3 Current and voltage phasors in circuits

These three properties translate circuit laws from time domain into phasor domain:

$$
\begin{array}{lccccr}
 & \textit{time} & \longleftrightarrow & \textit{phasor} & \\
(\textsc{kvl}) & \sum_{\text{loop}} v = 0 & \longleftrightarrow & \sum_{\text{loop}} V = 0 & (8.26)\\
(\textsc{kcl}) & \sum_{\text{node}} i = 0 & \longleftrightarrow & \sum_{\text{node}} I = 0 & (8.27)
\end{array}
$$

as well as current-voltage relationships:

$$
\begin{array}{lccccr}
\text{(resistor)} & v = Ri & \longleftrightarrow & V = RI & (8.28)\\
\text{(capacitor)} & i = C\frac{d}{dt}v & \longleftrightarrow & I = j\omega CV & (8.29)\\
\text{(inductor)} & v = L\frac{d}{dt}i & \longleftrightarrow & V = j\omega LI & (8.30)
\end{array}
$$

RC revisited

Because capacitors establish a proportional relationship between their voltage and current phasors, they may be regarded as impedances in phasor domain having impedance $\frac{1}{j\omega C}$. A sinusoidally-excited RC circuit is translated into phasor domain in Figure 8.1. In phasor domain, V_o is recognized as the lower half of a voltage divider spanning V_{in}:

$$V_o = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} V_{in} \tag{8.31}$$

$$= \frac{1}{1 + j\omega CR} V_{in} \tag{8.32}$$

Converting back to time domain,

$$v_o(t) = \text{Re}\left\{V_o e^{j\omega t}\right\} \tag{8.33}$$

$$= \text{Re}\left\{\frac{1}{1+j\omega CR} V_{in} e^{j\omega t}\right\} \tag{8.34}$$

Changing V_o to polar form,

$$= \text{Re}\left\{\frac{1}{|1+j\omega CR| e^{j\angle(1+j\omega CR)}} V_{in} e^{j\omega t}\right\} \tag{8.35}$$

$$= \text{Re}\left\{\frac{1}{|1+j\omega CR| e^{j\tan^{-1}(\omega CR)}} V_{in} e^{j\omega t}\right\} \tag{8.36}$$

$$= \text{Re}\left\{\frac{V_{in}}{|1+j\omega CR|} e^{j\left(\omega t - \tan^{-1}(\omega CR)\right)}\right\} \tag{8.37}$$

$$v_o(t) = \frac{V_{in}}{|1+j\omega CR|} \cos\left(\omega t - \tan^{-1}(\omega CR)\right) \tag{8.38}$$

For low frequencies ($\omega RC \ll 1$), $|1 + j\omega CR|$. The output has about the same amplitude as the input. For high frequencies ($\omega RC \ll 1$),[1] $|1 + j\omega CR|$. As the amplitude at the filter output vanishes at high frequencies, this RC circuit functions as a low-pass filter.

[1] The approximation is good when $\omega RC > 10$.

Lecture 9

Frequency Response and Bode Plots

9.1 Phasors review

Phasors analyze a system at a single frequency ω. Because the period of a complex exponential is 2π, ω is naturally expressed in rad/s. Conversion to cycles per second (f, in Hz) is given by $\omega = 2\pi f$, and the period is $T = \frac{1}{f}$.

In the differential equation

$$\frac{d}{dt}\vec{x} = A\vec{x}(t) + \vec{b}u(t) \tag{9.1}$$

with sinusoidal input $u(t)$, phasor analysis can lead to a particular solution for $x(t)$ with sinusoidal components.

The following identities relate complex numbers to sinusoids:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t \tag{9.2}$$

$$\cos\omega t = \text{Re}\left[e^{j\omega t}\right] = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \tag{9.3}$$

$$\sin\omega t = \text{Im}\left[e^{j\omega t}\right] = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \tag{9.4}$$

Phasors respect the following properties when translating to and from time domain:

uniqueness There is a one-to-one correspondence between functions $A\cos(\omega t + \phi)$ and phasors $Ae^{j\phi}$, where A is real and positive.

linearity If a_1 and a_2 are real numbers, then the following addition law holds vertically:

$$a_1 x_1(t) = A_1\cos(\omega t + \phi_1) \longleftrightarrow A_1 e^{j\phi_1} \tag{9.5}$$

$$a_2 x_2(t) = A_2\cos(\omega t + \phi_2) \longleftrightarrow A_2 e^{j\phi_2} \tag{9.6}$$

$$a_1 x_1(t) + a_2 x_2(t) = \ldots \longleftrightarrow A_1 e^{j\phi_1} + A_2 e^{j\phi_2} \tag{9.7}$$

differentiation Differentiation in time domain corresponds to multiplication by $j\omega$ in phasor domain.

$$x(t) \longleftrightarrow Ae^{j\phi} \tag{9.8}$$

$$\frac{d}{dt}x(t) \longleftrightarrow j\omega Ae^{j\phi} \tag{9.9}$$

In a vector-valued system excited by sinusoidal input u,

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t), \tag{9.10}$$

Let $\vec{X}$ be a (vector) phasor representing $\vec{x}$:

$$\vec{x}(t) = \overrightarrow{\mathrm{Re}}\left[\vec{X}e^{j\omega t}\right], \tag{9.11}$$

and U a (scalar) phasor representing u:

$$u(t) = \mathrm{Re}\left[e^{j\omega t}U\right], \tag{9.12}$$

so that Equation 9.10 translates,

$$j\omega\vec{X} = A\vec{X} + \vec{b}U. \tag{9.13}$$

The particular solution has phasor X.

$$\vec{X} = (j\omega I - A)^{-1}\vec{b}U \tag{9.14}$$

Because of the linearity property, linear laws such as KVL (a linear relationship of branch voltages) and KCL (a linear relationship of branch currents) apply to phasor voltages and currents, respectively.

Resistors are linear circuit elements, with a proportionality relationship between voltage and current that holds in phasor domain:

$$v = Ri \qquad \longleftrightarrow \qquad V = RI \tag{9.15}$$

Capacitors are also linear circuit elements, however the current-voltage proportionality in phasor domain has an imaginary ratio.

$$i = C\frac{d}{dt}v \quad \longleftrightarrow \quad I = \underbrace{j\omega C}_{\text{admittance}} V \tag{9.16}$$

The proportionality $i = Gv$ (time domain, G real) is called conductance. In phasor domain with a complex ratio, it is called *admittance*. The inverse of admittance is called *impedance*, and generalizes resistance.

$$i = C\frac{d}{dt}v \quad \longleftrightarrow \quad V = \underbrace{\frac{1}{j\omega C}}_{\text{cap. impedance}} I \tag{9.17}$$

$$v = L\frac{d}{dt}i \quad \longleftrightarrow \quad V = \underbrace{j\omega L}_{\text{ind. impedance}} I \tag{9.18}$$

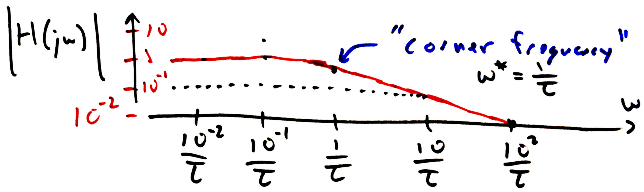

Figure 9.1: Bode magnitude plot of Equation 9.19.

9.2 Transfer functions

We can now solve for particular solutions algebraically. From the RC filter example, Equation 8.32 can be solved for a phasor ratio V_o/V_{in}:

$$H(j\omega) = \frac{V_o}{V_{in}} = \frac{1}{1 + j\omega RC}. \tag{9.19}$$

$H(j\omega)$ is called the *transfer function* of this system. It is a complex-valued function of angular frequency ω whose magnitude is the amplitude scaling factor of this input-output signal pair, and whose phase is the phase shift. Analyzing systems by following algebraic functions of a frequency parameter ω is a strategy generally called "frequency domain."[1]

Bode plots

Engineers like to examine information at log scales, especially when it spans orders of magnitude.[2] Frequency perception in human hearing ranges from roughly 20 Hz to 20 kHz.

A *Bode plot* of the transfer function $H(j\omega)$ is the following two things:

1. A log-log plot of $|H(j\omega)|$ vs. ω.
2. A angle-log plot of $\angle H(j\omega)$ vs. ω.

Magnitude

For $H(j\omega)$ above,

$$|H(j\omega)| = \left|\frac{1}{1 + j\omega RC}\right| \tag{9.20}$$

$$= \frac{1}{\sqrt{1 + \omega^2 (RC)^2}} \tag{9.21}$$

[1] An ideal hi-fi audio system, for example, would convert data on the recording medium to sound pressure in the air in a way that treats all frequencies equally.

[2] A piano keyboard lays out fundamental frequencies from left to right on a log scale.

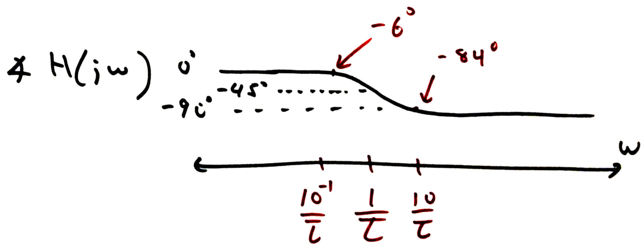

Figure 9.2: Bode phase plot of Equation 9.19.

On a log-log scale (Figure 9.1), this looks approximately like a flat left asymptote and a downhill right asymptote, meeting at $H(j\omega)\big|_{\omega=\frac{1}{\tau}} \approx 1$.[3]

Phase

To find the phase of $H(j\omega)$, we can write it as a complex number in rectangular form, times a real coefficient:

$$H(j\omega) = \frac{1}{1 + j\omega RC} = \frac{1 - j\omega RC}{1 + \omega^2 (RC)^2} = \frac{1 - j\omega\tau}{(\textit{positive real})} \tag{9.22}$$

The numerator is in the fourth quadrant of the complex plane, and its angle of depression is given by

$$\theta = \tan^{-1}\left(\frac{\text{rise}}{\text{run}}\right) \tag{9.23}$$

$$= \tan^{-1}\left(\frac{-\omega\tau}{1}\right) = \tan^{-1}(-\omega\tau) \tag{9.24}$$

$$= -\tan^{-1}(\omega\tau)\,. \tag{9.25}$$

For positive inputs on a log scale, the inverse tangent function smoothly transitions from 0 to 90 degrees, crossing 45 degrees at 1. Our function, plotted in Figure 9.2, is the opposite. It has a left asymptote of 0 degrees and a right asyptote of -90 degrees. The transition, centered at $\omega^* = \frac{1}{\tau}$, is so fast within a multiple by 10 of ω that the asymptotes look nearly flat left and right of the transition region.

[3]It's actually $\frac{1}{\sqrt{2}}$ (a real number) but this approximation works better with the piecewise linear style.

Lecture 10

Resonance in RLC Circuits

To analyze the LRC circuit in Figure 10.1, assign phasors to both v_{in} and v_o:

$$v_{in}(t) = \text{Re}\left[\underbrace{V_{in}}_{\text{phasor}} e^{j\omega t}\right] \tag{10.1}$$

$$v_o(t) = \text{Re}\left[\underbrace{V_o}_{\text{also phasor}} e^{j\omega t}\right] \tag{10.2}$$

V_o spans the rightmost leg of a a three-way voltage divider (V_{in} among impedances $j\omega L$, R, and $\frac{1}{j\omega C}$, so the transfer function is the following impedance ratio:

$$\frac{V_o}{V_{in}} = H(j\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R + j\omega L} \tag{10.3}$$

$$= \frac{1}{LC\,(j\omega)^2 + j\omega RC + 1} \tag{10.4}$$

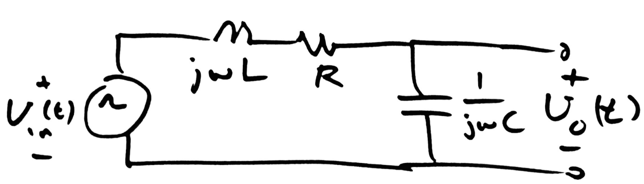

Figure 10.1: An LRC circuit.

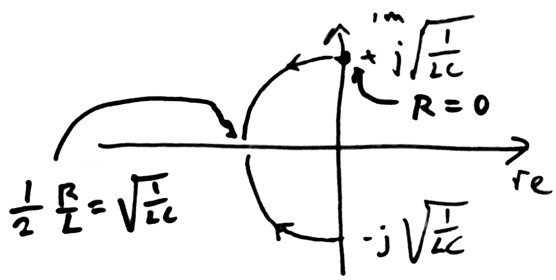

Figure 10.2: Eigenvalue locus from imaginary pair at $R = 0$ to negative real at critical R, as R increases from 0.

10.1 Time-domain analysis

Eigenvalues, two ways

In Equation 7.25, we found a state space differential equation model of this circuit with the following matrix:

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \tag{10.5}$$

Its eigenvalues,

$$\lambda_{1,2} = -\frac{1}{2}\frac{R}{L} \pm j\sqrt{\left(\frac{1}{2}\frac{R}{L}\right)^2 - \frac{1}{LC}} \tag{10.6}$$

are a complex conjugate pair on the imaginary axis when $R = 0$. As R approches a critical value, they slowly approach the same point on the negative real axis. Their rendezvous is depicted in Figure 10.2. (Afterwards, they separate but remain real and negative.) We will reparameterize these eigenvalues in a way that traces their trajectory. Call the undamped (angular) frequency ω_n:

$$\lambda_{1,2}\Big|_{R=0} = \pm\sqrt{-\frac{1}{LC}} = \pm j\omega_n, \tag{10.7}$$

and define a damping coefficient ξ (Greek letter xi) that goes from 0 to 1 as the two eigenvalues leave the imaginary axis and meet at a negative real.

$$\xi = \frac{1}{2}\frac{R}{\sqrt{\frac{L}{C}}} \tag{10.8}$$

This parameterization appears in Figure 10.3.

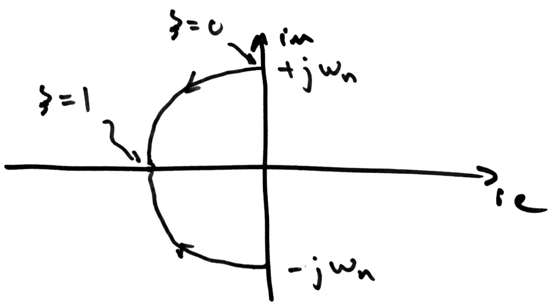

Figure 10.3: Figure 10.1, but reparameterized using ω_n and ξ.

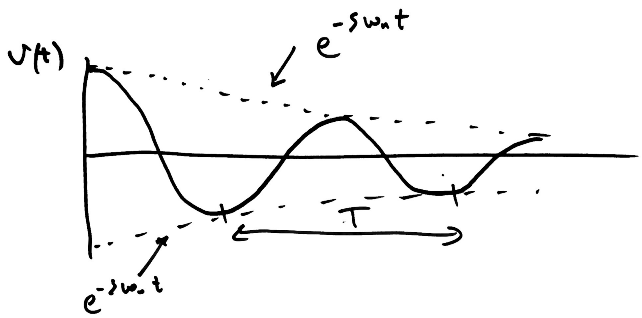

Figure 10.4: Homogeneous response of an LRC circuit with $R > 0$.

Homogeneous response

With the same choices of ω_n and ξ, the eigenvalues of A can be expresssed as

$$\lambda_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}. \tag{10.9}$$

The general form of the homogeneous response (modulo possible scaling and phase shift) is

$$v(t) = \text{Re}\left\{e^{\lambda_1 t}\right\} \tag{10.10}$$

$$= e^{-\xi\omega_n t}\,\text{Re}\left\{e^{j\omega_n\sqrt{1-\xi^2}t}\right\} \tag{10.11}$$

$$= e^{-\xi\omega_n t}\cos\left(\omega_n\sqrt{1-\xi^2}t\right) \tag{10.12}$$

The graph of $v(t)$, sketched in Figure 10.4, is a sinusoid with period $\frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$, trapped inside an exponential decaying at rate $\xi\omega_n$.

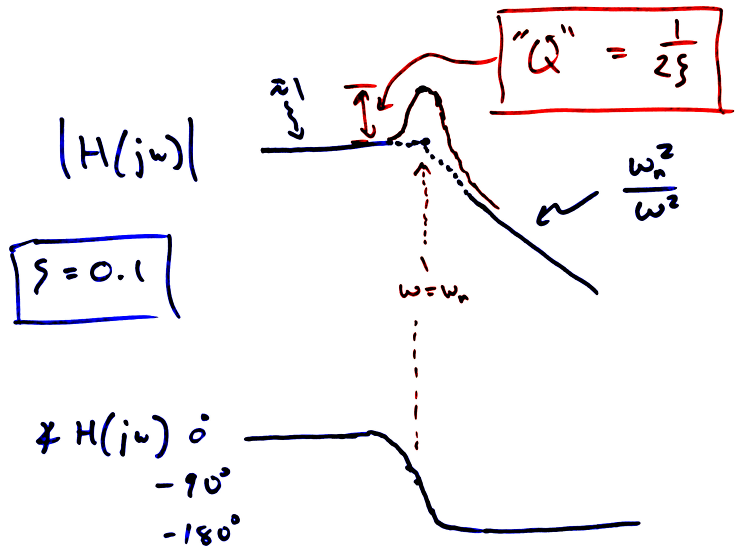

Figure 10.5: Bode plot of an LRC filter with $\xi = 0.1$.

10.2 Reparameterized transfer function

With the ω_n-ξ parameterization, this circuit's transfer function becomes

$$\frac{V_o}{V_{in}} = H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + (j\omega)\, 2\xi\omega_n + \omega_n^2} \tag{10.13}$$

We can imagine the Bode plot of $H(j\omega)$ as having three pieces:

- When $\omega \ll \omega_n$, $H(j\omega) \approx 1$.
- When $\omega = \omega_n$, $H(j\omega) = \frac{-j}{2\xi}$. The magnitude is called "Q."[1]
- When $\omega \gg \omega_n$, $H(j\omega) \approx \frac{-\omega_n^2}{\omega^2}$.

A possible plot is shown in Figure 10.5.

10.3 Applications of (R)LC filtering

Radio

The power amplifier in a cellular handset runs off a low voltage, limited by the typical single cell lithium-ion battery voltage of about 3.6 V. Internally, the amplifier circuit is only able to generate a sinusoid of voltage in the 1–2 V range. This would create an extreme challenge in driving an antenna with

[1]Quality factor, or how selective this filter is of its favorite frequency.

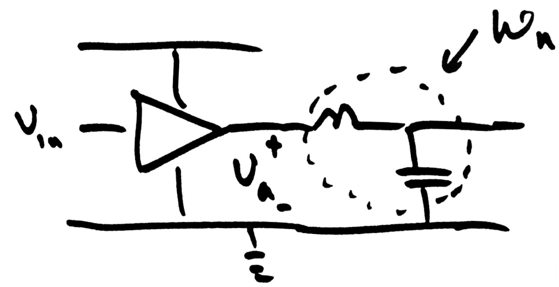

Figure 10.6: An $R \approx 0$ LC circuit used as an matching network.

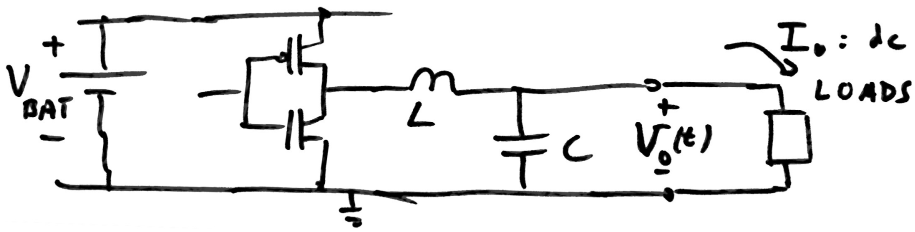

Figure 10.7: An $R \approx 0$ LC circuit used in a DC-DC converter.

impedance in the 50–75 Ω range; namely, it would be impossible to develop the ≈ 1 W power level without somehow boosting the voltage between the solid-state amplifier and the antenna. This is exactly where the LC network of Figure 10.6 comes to the rescue. In this example, the LC network is called a matching network, and is used to boist the voltage to better "match" the antenna impedance. A matching network as in Figure 10.6 before the antenna improves the performance by altering the effective impedance of the antenna.

$$v_{in} = \underbrace{a(t)}_{\text{slow}} \cos\left(\underbrace{\omega_n t}_{\text{fast}} + \underbrace{\phi(t)}_{\text{slow}}\right) \tag{10.14}$$

Taking the capacitor voltage as an output, we have $|v_o| \approx \frac{1}{2\xi}|v_a|$.

DC-DC converter

To step down a DC voltage source by 50%, we can use an inverter operating alternating at a 50% duty cycle to generate an offset square wave whose DC component is the voltage we are trying to deliver.[2] An LC filter reduces the AC component of the switch output, without dissipating power.

[2]For efficiency, we can use parallel NMOS and PMOS transistors to realize the switch to reduce resistance. We also want an inductor with the smallest possible parasitic resistance.

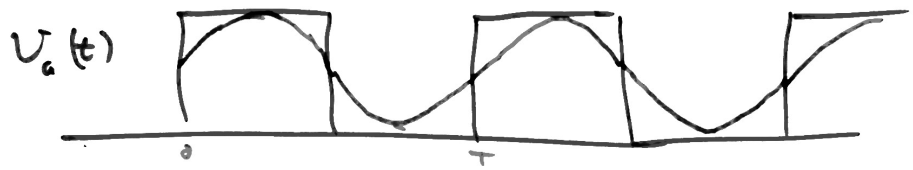

Figure 10.8: Output of the switch in Figure 10.7, approximated as an offset sine wave.

The inverter output v_a can be approximated as a sine wave with a DC offset (Figure 10.8):

$$v_a(t) = \underbrace{\frac{1}{2}V_{bat}\sin(\omega t)}_{ac} + \underbrace{\frac{1}{2}V_{bat}}_{dc} \tag{10.15}$$

We can use superposition to compute v_o. The filter passes DC at unity gain, and it scales and shifts the AC component.

$$v_o(t) = \frac{1}{2}V_{bat} + V_o\cos(\omega t + \phi) \tag{10.16}$$

In order to have a convincing DC output, we need $V_o \ll V_{bat}$. That sinusoid, as well as everything else that makes a square wave a square wave, will have to fit into the $|H(j\omega)| \approx \frac{\omega_n^2}{\omega^2}$ high-frequency region of Equation 10.13. That means we need to choose an ω_n such that $\omega_n \ll \omega$. For the DC amplitude to exceed the AC "ripple" amplitude by a factor of 100, for instance, ω would have to exceed $10\omega_n$.