Final Exam Review

1. Potpourri (Spring 2021 Final)

(a)

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{1}$$

Run Gram-Schmidt on these vectors in this order (that is, start with $\vec{v}_1$ then $\vec{v}_2$), and extend this set to form an orthonormal basis for $\mathbb{R}^3$. Show your work.

2. Analyzing an LC-LC Band-Stop/Notch Filter (Spring 2021 Midterm)

In this sub-part, you will partially analyze a circuit built entirely out of L, C components as shown in Figure 1. Assume that the circuit is operating at a frequency of $\omega = \omega_s$ (i.e. $v_{\text{in}}(t) = \cos(\omega_s t)$).

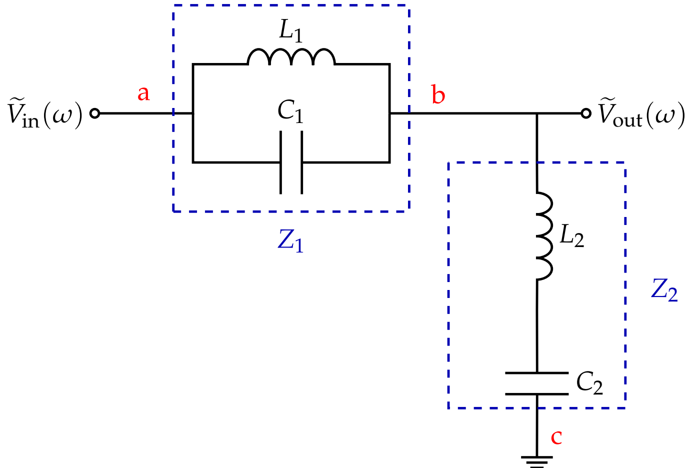

Figure 1: LC bandstop filter.

(a) **Find $\widetilde{V}_{\text{out}}(\omega)$ in terms of Z_1, Z_2, $\widetilde{V}_{\text{in}}(\omega)$.** *You do not need to compute $\widetilde{V}_{\text{in}}(\omega)$ for this part.* Show your work.

(b) **Find Z_1, the equivalent impedance between terminals a and b, in terms of L_1, C_1, and ω_s.** Leave your answer in the form $j\frac{M}{N}$, where M and N are real.

What is the impedance Z_1 at $\omega_s = \frac{1}{\sqrt{L_1 C_1}}$

(c) **Find Z_2, the equivalent impedance between terminals b and c, in terms of L_2, C_2, and ω_s.** Leave your answer in the form j$\frac{M}{N}$, where M and N are real.

What is the impedance Z_2 at $\omega_s = \frac{1}{\sqrt{L_2 C_2}}$?

3. Control (Fall 2019 Midterm 2)

Suppose that we have a two-dimensional discrete-time system governed by:

$$\vec{x}(t+1) = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{7}{6} \end{bmatrix} \vec{x}(t) + \vec{w}(t) \tag{2}$$

(a) **Is the system stable? Why or why not?**

Here, we give you that

$$A = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{7}{6} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{4}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \tag{3}$$

and that the characteristic polynomial $\det(\lambda I - A) = \lambda^2 + \frac{11}{6}\lambda + \frac{2}{3}$.

(b) Suppose that there is no disturbance and we can now influence the system using a scalar input $u(t)$ to our system:

$$\vec{x}(t+1) = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{7}{6} \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{4}$$

Is the system controllable?

(c) We want to set the closed-loop eigenvalues of the system

$$\vec{x}(t+1) = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{7}{6} \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{5}$$

to be $\lambda_1 = -\frac{5}{6}, \lambda_2 = \frac{5}{6}$ using state feedback

$$u(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}(t) \tag{6}$$

What specific numeric values of k_1 and k_2 should we use?

4. Controllable Canonical Form (Fall 2019 Final)

Suppose that we have a two-dimensional continuous time system governed by:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 1 & -1 \\ 0 & -4 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{7}$$

We would like to put this system into Controllable Canonical Form (CCF) to use state feedback to place the eigenvalues at desired locations. For your convenience, the characteristic polynomial is

$$\det\left(\lambda I - \begin{bmatrix} 1 & -1 \\ 0 & -4 \end{bmatrix}\right) = (\lambda - 1)(\lambda + 4) = \lambda^2 + 3\lambda - 4 \tag{8}$$

(a) **Is the system stable? Why or why not?**

(b) Recall that our original system is:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 1 & -1 \\ 0 & -4 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{9}$$

We would like to change the coordinates to bring the system into CCF:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{10}$$

Compute the T basis such that $\vec{z}(t) = T^{-1}\vec{x}(t)$ or equivalently (if you want), give us T^{-1} directly. To help you along, here are some calculations already done for you:

$$\begin{bmatrix} 1 & 0 \\ 1 & -4 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 1 & -1 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix}^{-1} = \begin{bmatrix} -a & 1 \\ 1 & 0 \end{bmatrix} \tag{12}$$

What is a_0? What is a_1?

(c) Using state feedback

$$u(t) = \begin{bmatrix} \widetilde{k}_0 & \widetilde{k}_1 \end{bmatrix} \vec{z}(t) \tag{13}$$

place the closed-loop eigenvalues at $\lambda_1 = -1$, $\lambda_2 = -2$. **What is $\widetilde{k}_0$? What is $\widetilde{k}_1$?**

(Notice that we are asking for the feedback gains in terms of $\vec{z}(t)$ not the original $\vec{x}(t)$. If you have time, feel free to check your work by the original $\vec{x}(t)$.)

5. I bet Cal will win this year (Spring 2021 Final)

As huge fans of the Big Game, you and your friend want to bet on whether Cal or Stanford will win this year. You want to predict this year's result by analyzing historical records. Therefore, you decide to model this as a binary classification problem and do PCA for dimension reduction on the data you collected. The "+1" class represents victories of Cal and "−1" represents victories of Stanford.

After some research, you obtained a data matrix $A \in \mathbb{R}^{n \times d}$,

$$A = \begin{bmatrix} - & \vec{x}_1^\top & - \\ - & \vec{x}_2^\top & - \\ & \vdots & \\ - & \vec{x}_n^\top & - \end{bmatrix} \tag{14}$$

where each of the n rows $\vec{x}_i^\top$ denotes a game and each of the d columns of A contains information of a possibly relevant factor of the games (weather, location, date, air quality, etc).

(a) Let the full SVD of $A = U\Sigma V^\top$, where A is given in eq. (14).

You project your data along $\vec{v}_1$ and $\vec{v}_2$ (the first two principal components along the rows), and for comparison you also project your data along two randomly chosen directions $\vec{w}_1$ and $\vec{w}_2$ as well. You get the two pictures in Figure 2, but you forgot to label the axes. Of the two figures below, which one is the projection onto the principal components and which one is the projection onto the random directions? **Match axes (i), (ii), (iii), (iv) to $\vec{w}_1, \vec{w}_2, \vec{v}_1$, and $\vec{v}_2$, and justify your answer.**

Note that there may be multiple correct matchings; you only need to find and justify one of them.

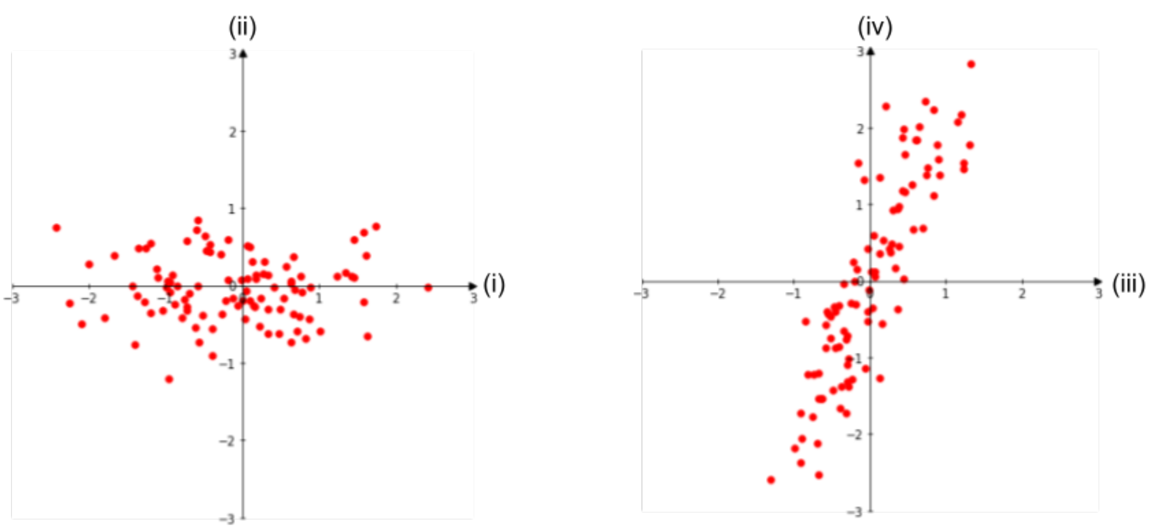

Figure 2: Projected datasets.

(b) In order to reduce the dimension of the data, we would like to project the data onto the first k principal components along the rows of A, where k is less than the original data dimension d. **Show how to find the new coordinates $\vec{z}_i$ of the data point $\vec{x}_i$ after this projection.** You may use the SVD of A.

(c) Using the data you have, you trained a classifier $\vec{w}_\star$. For any new data point after dimension reduction $\vec{z}_{\text{new}}$, the value of $\text{sign}(\vec{w}_\star^\top \vec{z}_{\text{new}})$ tells you whether the data point belongs to the "+1" class or to the "−1" class. Now suppose you have obtained two new data points, $\vec{z}_a$ and $\vec{z}_b$. Based on Figure 3 showing $\vec{w}_\star$, $\vec{z}_a$ and $\vec{z}_b$, **predict the class of $\vec{z}_a$ and $\vec{z}_b$ using $\vec{w}_\star$, and justify your answer.**

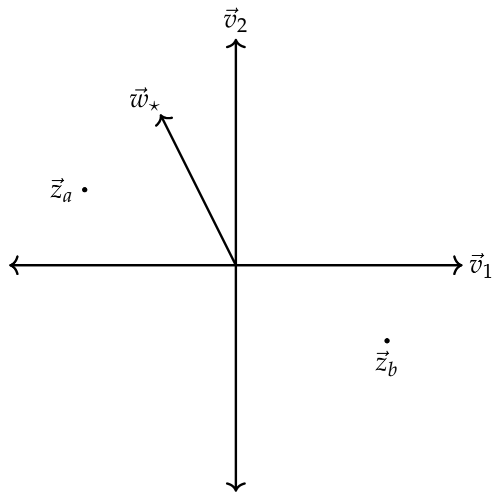

Figure 3: Dataset projected onto $\vec{v}_1$ and $\vec{v}_2$ with $\vec{w}_\star$

(d) Assume $d = 6$, $k = 4$, and $\vec{w}_\star = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^\top$. Let $A = U\Sigma V^\top$ for A defined in eq. (14), and you find that V is given by the identity matrix, i.e. $V = I_d$. Now suppose the data point for this year's big game $\vec{x}_{2021} = \begin{bmatrix} 3 & 6 & 4 & 1 & 9 & 6 \end{bmatrix}^\top$. **Would you bet on Cal or Stanford to win? Justify your answer.** A quick reminder that "+1" denotes victories of Cal and "−1" denotes victories of Stanford.

(HINT: Don't forget to project your data onto the principal components.)

6. Optimization and Singular Values (Spring 2021 Final)

We are going to focus on a special optimization problem that is related to the underlying structure of the SVD. More specifically, we want to solve for s in the following maximization problem

$$s = \max_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|^2}{\|\vec{x}\|^2} \tag{15}$$

Here, we have $A \in \mathbb{R}^{m \times n}$. Let $m > n$ so that A is a tall matrix and $\text{rank}(A) = n$. Let the full SVD be given by $A = U\Sigma V^\top$. Define $\vec{x}^\star \in \mathbb{R}^n$ to be the optimal vector that achieves the maximum in equation (15). That is,

$$\vec{x}^\star = \underset{\|\vec{x}\| \neq 0}{\text{argmax}} \frac{\|A\vec{x}\|^2}{\|\vec{x}\|^2} \tag{16}$$

$$s = \frac{\left\|A\vec{x^\star}\right\|^2}{\left\|\vec{x^\star}\right\|^2} \tag{17}$$

(a) We start by attempting to simplify the optimization problem. **Prove that for any $\vec{x}$, we have $\|A\vec{x}\| = \left\|\Sigma V^\top \vec{x}\right\|$.** Note that you must justify and explain every step for full credit, just equations without an explanation may not be awarded full credit.

(b) Using a change of variables, we can in fact turn our original maximization problem into

$$s = \max_{\|\vec{w}\| \neq 0} \frac{\|\Sigma\vec{w}\|^2}{\|\vec{w}\|^2}. \tag{18}$$

Find the correct change of variables that relates $\vec{x}$ and $\vec{w}$ and show that optimization problems (15) and (18) are equivalent.

(HINT: The change of variables you are looking for can also be thought of as a change of basis.)

(c) **Let σ_1 be the largest singular value of matrix A. Find a $\vec{w}^\star$, such that $\|\Sigma\vec{w}^\star\|^2 = \sigma_1^2\|\vec{w}^\star\|^2$. Justify your answer.**

(d) **Prove that for all $\vec{w}$ we have $\|\Sigma\vec{w}\|^2 \leq \sigma_1^2\|\vec{w}\|^2$. Show your work.**