

1. Honor Code (0 pts.)

Please copy the following statement in the space provided below and sign your name.

As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

Solution: Any attempt to copy the honor code and sign should get full points.

2. What are you planning to do after the midterm? (2 pts.)

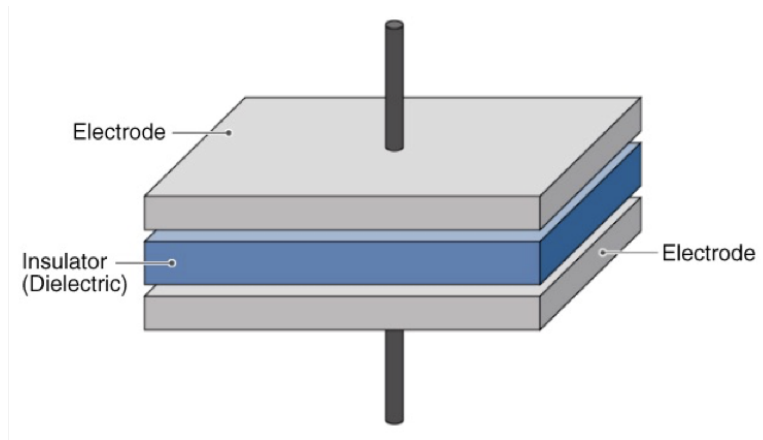
Solution: Any answer is sufficient.

3. What's your favorite thing to do in Berkeley? (2 pts.)

Solution: Any answer is sufficient.

4. Capacitor and Inductor Basics (20 pts.)

- (a) (4 pts.) **What is the capacitance of the following parallel plate capacitor?** The permittivity of the material is known to be $10^{-10} \frac{\text{F}}{\text{m}}$, the dimensions of the larger face of the plate is $2 \mu\text{m}$ by $3 \mu\text{m}$, and the thickness of the insulator is $1 \mu\text{m}$.



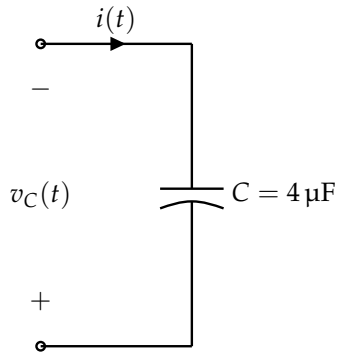
Solution: The capacitance is given by

$$C = \epsilon \frac{A}{d} = 10^{-10} \frac{6 \times 10^{-12}}{1 \times 10^{-6}} = 6 \times 10^{-16} \text{F} \quad (1)$$

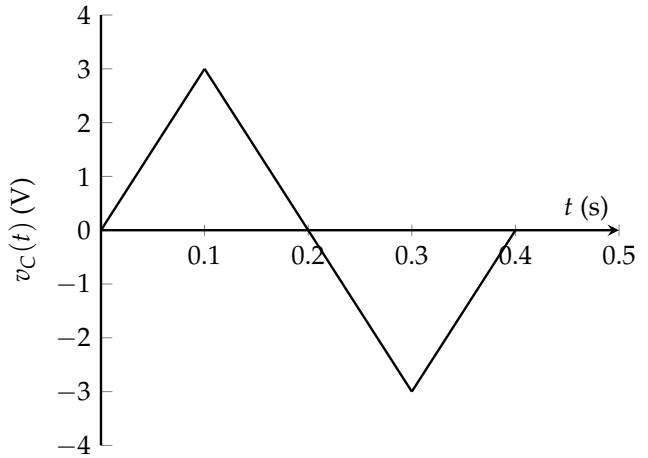
(b) (8 pts.) A capacitance and its voltage with respect to time is shown below. **Qualitatively sketch the**

- i. **Current**
- ii. **Power**

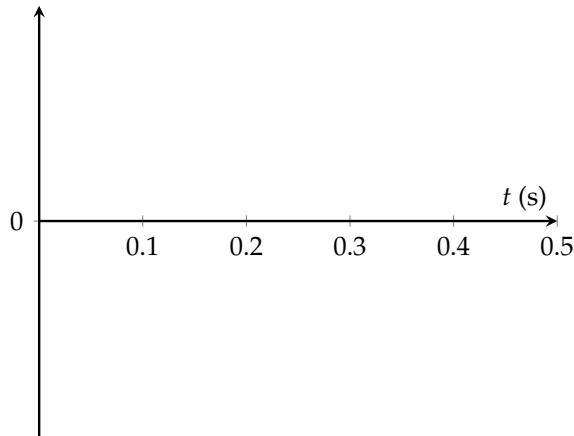
as functions of time. When you are qualitatively sketching, you may define your own axes and scaling. The sketches you make do not have to be to scale. **The important part is to note the general shape of the curve – you do not have to denote exact values on your coordinate axes.** Feel free to use the blank axes below.

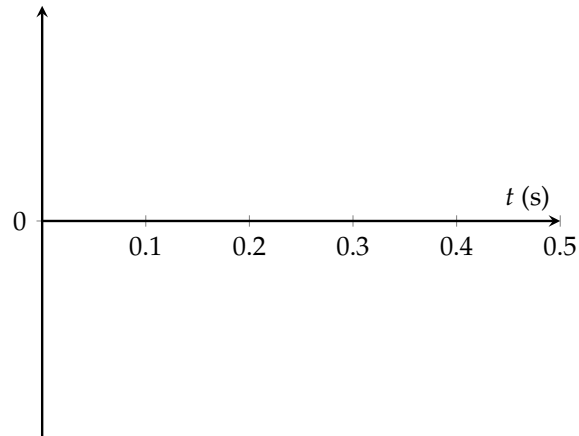


(a) Capacitor Circuit

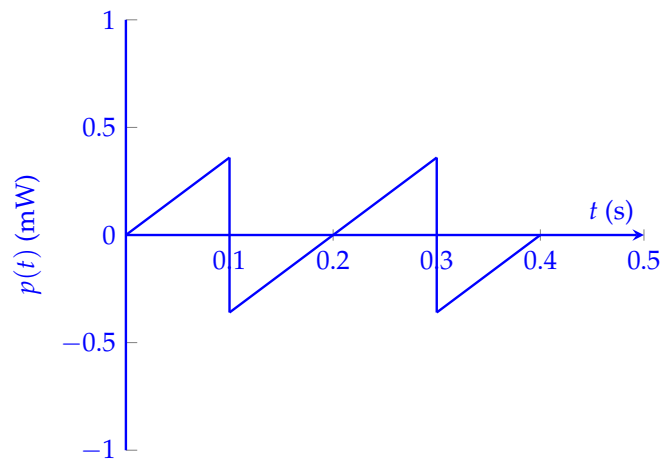
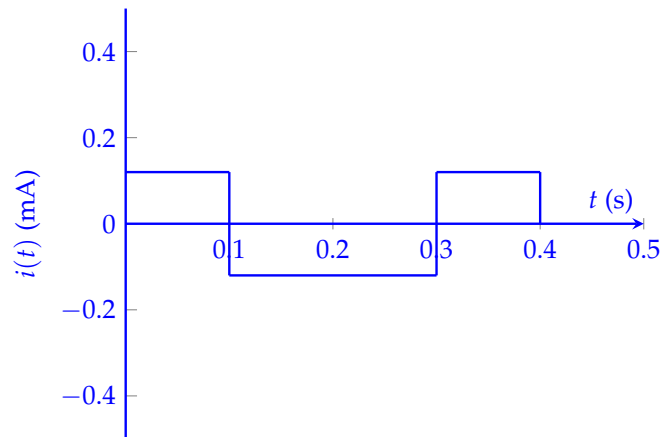


(b) Plot of $v_C(t)$





Solution: The shapes of the plots of current and power should roughly resemble the plots below, respectively:



Mathematical Derivation of Plots (OPTIONAL):

We can write the voltage as a piecewise function of time, namely:

$$v_C(t) = \begin{cases} 30t & 0 \leq t \leq 0.1 \\ -30t + 6 & 0.1 \leq t \leq 0.3 \\ 30t - 12 & 0.3 \leq t \leq 0.4 \end{cases} \quad (2)$$

Using this, we can compute $i(t) = C \frac{dv_C(t)}{dt}$ which is written as follows

$$i(t) = \begin{cases} 30 \times 4 \times 10^{-6} = 0.12 \times 10^{-3} & 0 \leq t \leq 0.1 \\ -30 \times 4 \times 10^{-6} = -0.12 \times 10^{-3} & 0.1 \leq t \leq 0.3 \\ 30 \times 4 \times 10^{-6} = 0.12 \times 10^{-3} & 0.3 \leq t \leq 0.4 \end{cases} \quad (3)$$

To find the power, we pointwise multiply $i(t)$ and $v_C(t)$ which yields

$$p(t) = \begin{cases} 3.6t & 0 \leq t \leq 0.1 \\ 3.6t - 0.72 & 0.1 \leq t \leq 0.3 \\ 3.6t - 1.44 & 0.3 \leq t \leq 0.4 \end{cases} \quad (4)$$

- (c) (8 pts.) At $t = 0$, an inductor $L = 1 \mu\text{H}$ which was carrying a current of 1 mA is connected across a resistor, $R = 1 \text{ k}\Omega$. Find the expression for current through the inductor as a function of time, and find the total energy that will be dissipated through the resistor.

(HINT: Remember, $E = \int P dt$.)

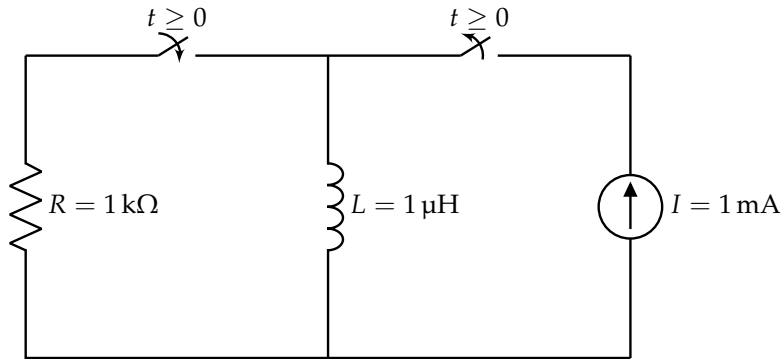


Figure 2: Sample Inductor Circuit

Solution: We know that the voltage drop across the inductor will be the negative of the voltage drop across the resistor (by KVL), so $v_L(t) = -i(t)R$. Furthermore, we know that, for an inductor, $\frac{di(t)}{dt} = \frac{v_L(t)}{L}$, so we have

$$\frac{di(t)}{dt} = -i(t)\frac{R}{L} \quad (5)$$

$$\frac{di(t)}{i(t)} = -\frac{R}{L} dt \quad (6)$$

$$\int_{i(0)}^{i(t)} \frac{di}{i} = \int_0^t -\frac{R}{L} dt' \quad (7)$$

$$\ln\left(\frac{i(t)}{i(0)}\right) = -\frac{R}{L}t \quad (8)$$

$$i(t) = i(0)e^{-\frac{R}{L}t} = 10^{-3}e^{-10^9t} \quad (9)$$

The power dissipated by the resistor is $(i(t))^2R = 10^{-3}e^{-2 \times 10^9t}$. Integrating from $t = 0$ to ∞ , we get

$$E = \int_0^\infty (i(t))^2R dt = \int_0^\infty 10^{-3}e^{-2 \times 10^9t} dt = 10^{-3} \times 0.5 \times 10^{-9} = 0.5 \times 10^{-12} \text{ J} \quad (10)$$

5. The Inevitable RLC Circuit (20 pts.)

Suppose you are given the following RLC Circuit as shown in fig. 3. V_{DC} is some unknown DC voltage.

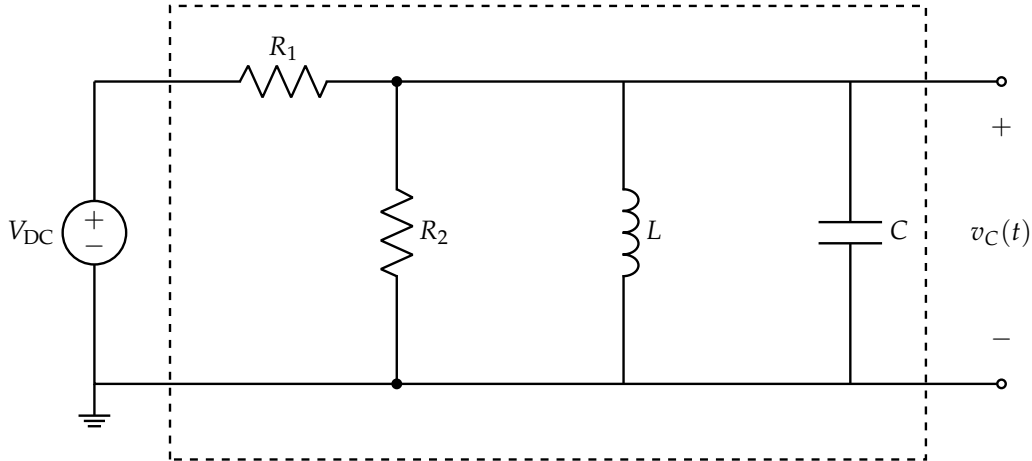


Figure 3

- (a) (8 pts.) Show that the differential equation that describes the voltage across the capacitor $v_C(t)$ as a function of time can be written as

$$\frac{d^2 v_C(t)}{dt^2} + M \frac{dv_C(t)}{dt} + N v_C(t) = 0 \tag{11}$$

where M and N are constants that may include $R_1, R_2, L,$ and C .

Solution: Suppose we label our circuit in the following way, following passive sign convention:

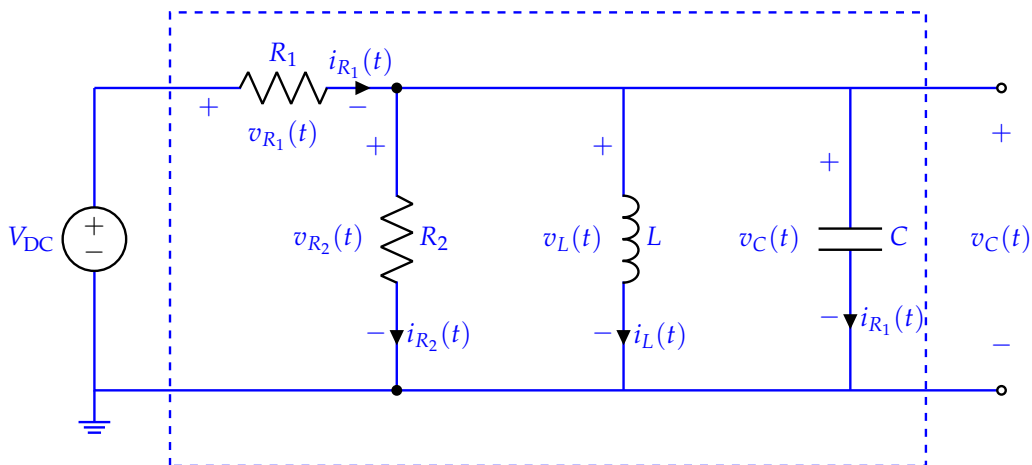


Figure 4: Labeled RLC Circuit

Writing out KCL, we have:

$$i_{R_2}(t) + i_L(t) + i_C(t) = i_{R_1}(t) \tag{12}$$

Writing the element equations, we get:

$$i_{R_1}(t) = \frac{v_{R_1}(t)}{R_1} \tag{13}$$

$$i_{R_2}(t) = \frac{v_{R_2}(t)}{R_2} \quad (14)$$

$$i_L(t) = \frac{1}{L} \int_{t_0}^t v_L(t') dt' + i_L(t_0) \quad (15)$$

$$i_C(t) = C \frac{dv_C(t)}{dt} \quad (16)$$

If we plug in nodal voltages into our equations, we get:

$$i_{R_1}(t) = \frac{V_{DC} - v_C(t)}{R_1} \quad (17)$$

$$i_{R_2}(t) = \frac{v_C(t)}{R_2} \quad (18)$$

$$i_L(t) = \frac{1}{L} \int_{t_0}^t v_C(t') dt' + i_L(t_0) \quad (19)$$

$$i_C(t) = C \frac{dv_C(t)}{dt} \quad (20)$$

Then substituting into our original KCL equation, we get:

$$\frac{V_{DC} - v_C(t)}{R_1} = \frac{v_C(t)}{R_2} + \frac{1}{L} \int_{t_0}^t v_C(t') dt' + i_L(t_0) + C \frac{dv_C(t)}{dt} \quad (21)$$

Then, we take the time derivative of the whole equation to get a pure differential equation. Then simplify to our "standard form":

$$\frac{d}{dt} \left[\frac{v_C(t)}{R_2} + \frac{1}{L} \int_{t_0}^t v_C(t') dt' + i_L(t_0) + C \frac{dv_C(t)}{dt} \right] = \frac{d}{dt} \left[\frac{V_{DC}}{R_1} - \frac{v_C(t)}{R_1} \right] \quad (22)$$

$$\frac{1}{R_2} \frac{dv_C(t)}{dt} + \frac{1}{L} v_C(t) + C \frac{d^2v_C(t)}{dt^2} = -\frac{1}{R_1} \frac{dv_C(t)}{dt} \quad (23)$$

$$C \frac{d^2v_C(t)}{dt^2} + \left(\frac{1}{R_2} + \frac{1}{R_1} \right) \frac{dv_C(t)}{dt} + \frac{1}{L} v_C(t) = 0 \quad (24)$$

$$\frac{d^2v_C(t)}{dt^2} + \underbrace{\left(\frac{R_1 + R_2}{CR_1R_2} \right)}_M \frac{dv_C(t)}{dt} + \underbrace{\frac{1}{LC}}_N v_C(t) = 0 \quad (25)$$

- (b) (8 pts.) Suppose that, if you were to plug in appropriate values of R_1 , R_2 , L , and C , you arrive at the following differential equation:

$$\frac{d^2 v_C(t)}{dt^2} + 10^9 \frac{dv_C(t)}{dt} + 10^{18} v_C(t) = 0 \quad (26)$$

Solve this differential equation to derive an expression for $v_C(t)$. Note that your solution should consider both homogeneous and particular solutions. The solution can also contain constants to be determined from initial conditions (no initial conditions have been specified for this problem).

What is the steady state solution for $v_C(t)$?

Solution: Here, we have a second order differential equation. We know that the solution for $v_C(t)$ will be the sum of the particular/steady state and homogeneous/complementary solutions:

$$v_C(t) = v_{C,p}(t) + v_{C,h}(t) \quad (27)$$

where $v_{C,p}(t)$ is the particular solution and $v_{C,h}(t)$ is the homogeneous solution.

To solve for the homogeneous solution, we will look at the values of our damping coefficient (α), damping ratio (ζ), and undamped resonant frequency (ω_0).

For the damping coefficient, we have that

$$2\alpha = \frac{R_1 + R_2}{CR_1R_2} \quad (28)$$

$$\alpha = \frac{2 \times 10^3 + 2 \times 10^3}{2(10^{-12})(2 \times 10^3)(2 \times 10^3)} \quad (29)$$

$$\alpha = 5 \times 10^8 \quad (30)$$

For the undamped resonant frequency, we get

$$\omega_0^2 = \frac{1}{LC} \quad (31)$$

$$\omega_0 = \sqrt{\frac{1}{10^{-6} \times 10^{-12}}} \quad (32)$$

$$\omega_0 = \sqrt{10^{18}} \quad (33)$$

$$\omega_0 = 10^9 \quad (34)$$

Since we have $\alpha < \omega$, our damping ratio is $\zeta = \frac{\alpha}{\omega_0} = \frac{5e8}{1e9} = 0.5$. Therefore, we have an underdamped system where our roots are complex. Thus, our complementary solution is in the form:

$$v_{C,h}(t) = K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t) \quad (35)$$

Solving for ω_n , which is the natural frequency, we have:

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2} \quad (36)$$

$$\omega_n = \sqrt{(10^9)^2 - (5 \times 10^8)^2} \quad (37)$$

$$\omega_n = \sqrt{10^{18} - 25 \times 10^{16}} \quad (38)$$

$$\omega_n = \sqrt{75 \times 10^{16}} \quad (39)$$

$$\omega_n = 5\sqrt{3} \times 10^8 \tag{40}$$

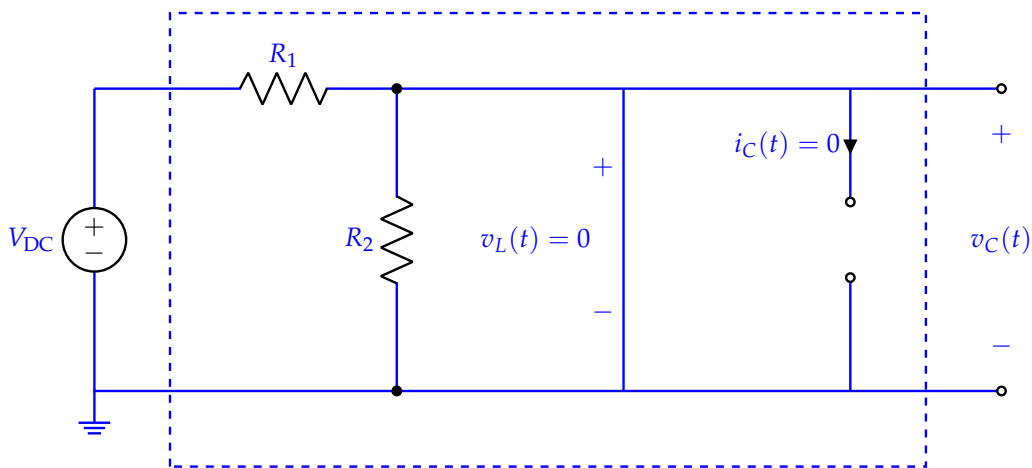
Therefore, our complementary solution can be written as

$$v_{C,h}(t) = K_1 e^{-5 \times 10^8 t} \cos(5\sqrt{3} \times 10^8 t) + K_2 e^{-5 \times 10^8 t} \sin(5\sqrt{3} \times 10^8 t) \tag{41}$$

where K_1 and K_2 are constants that can be solved for using initial conditions.

We can use DC steady state analysis to determine our particular solution. Recall that in DC steady state, our current and voltages are constant and therefore, their derivatives with respect to time are equal to 0. Thus, for a capacitor, we have $i_C(t) = C \frac{dv_C(t)}{dt} = 0$ A and can be treated as an open circuit, whereas an inductor we have $v_L(t) = L \frac{dv_L(t)}{dt} = 0$, meaning that we treat it as a short circuit or wire.

Redrawing our RLC Circuit, we have:



Notice that $v_C(t)$ is connected directly to ground and thus $v_{C,p}(t) = 0$. Therefore our general solution just the homogeneous solution.

$$v_C(t) = K_1 e^{-5 \times 10^8 t} \cos(\sqrt{75} \times 10^8 t) + K_2 e^{-5 \times 10^8 t} \sin(\sqrt{75} \times 10^8 t) \tag{42}$$

In steady state, we said that $v_C(t) = 0$, which we can confirm by taking the limit as $t \rightarrow \infty$ of Equation eq. (42), which will go to 0. Thus, in steady state:

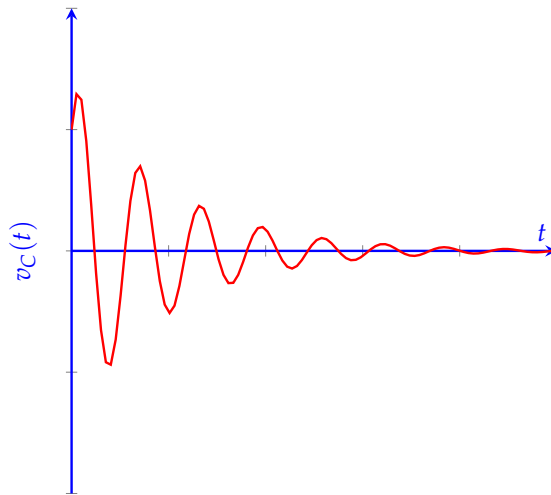
$$v_C(t) = 0 \tag{43}$$

- (c) (4 pts.) **Qualitatively sketch $v_C(t)$ as a function of time t .** When you are qualitatively sketching, you may define your own axes and scaling. The important part is to note the general shape of the curve – you do not have to denote exact values on your coordinate axes. In your qualitative plot, indicate whether the function is decaying or increasing with time, and make sure to note other important aspects of the function where applicable (e.g., oscillations, discontinuities, etc.).

Solution: We identified that the circuit is underdamped and therefore, we will have some oscillation that will decay over time to 0 which is our steady state solution. Since the initial condition wasn't specified, the shape of the graph and its asymptotic behavior are the only things we care about. The graph should contain the following two characteristics for full credit:

- i. Some decaying oscillation representing an underdamped circuit
- ii. A steady state of 0 V

Here is a sample solution that would receive full credit:



6. Let's Transfer to a Different Approach (15 pts.)

For this problem we will be using the same RLC circuit as the previous problem. It has been copied below for your convenience in fig. 5.

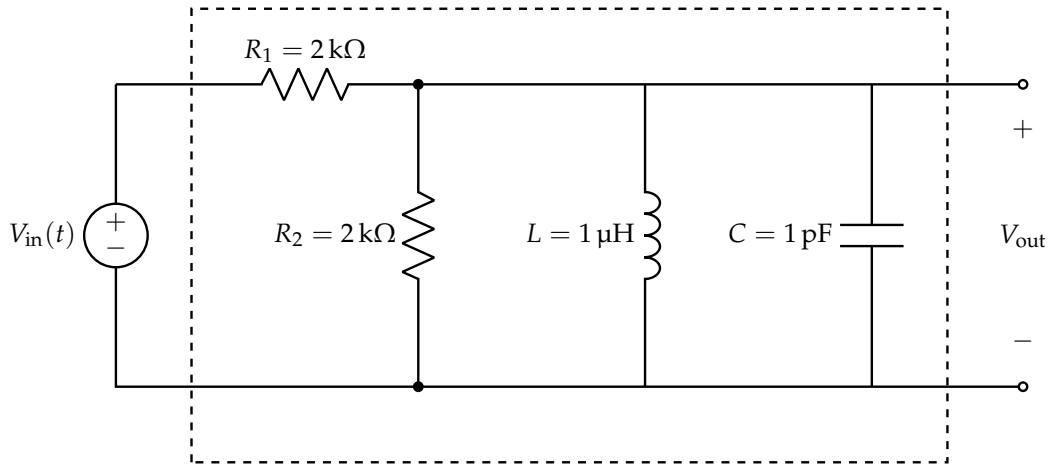


Figure 5

- (a) (8 pts.) For the circuit provided in fig. 5, **find the transfer function $H(f)$ considering the two port network indicated by the box in terms of R_1, R_2, L, C, f .** Put your answer in the form

$$H(f) = \frac{1}{k_1 + jk_2\left(\frac{f}{f_0} - \frac{f_0}{f}\right)} \quad (44)$$

where k_1 and k_2 are some real constants in terms of R_1, R_2, L, C and $\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}}$.

Solution: Let's start by converting the circuit into phasor domain, where sources are represented as phasors and components represented by their impedances.

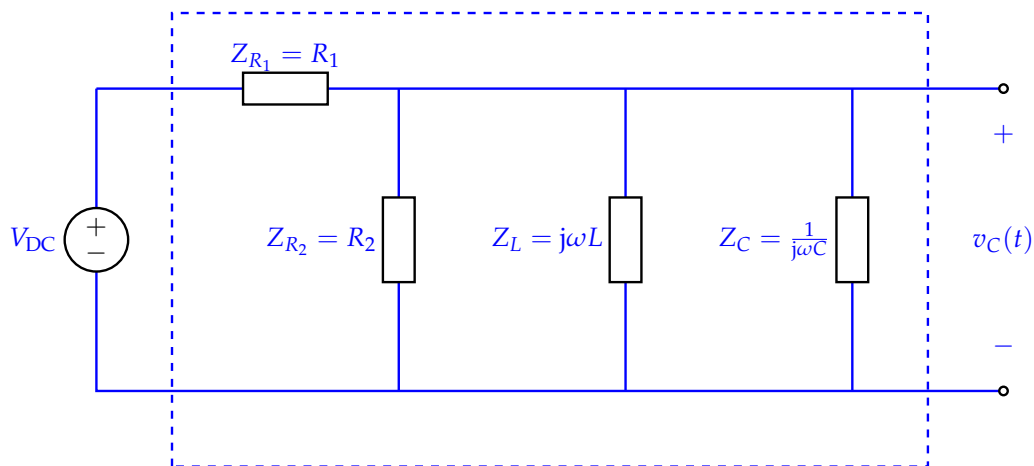


Figure 6

The circuit can be simplified as

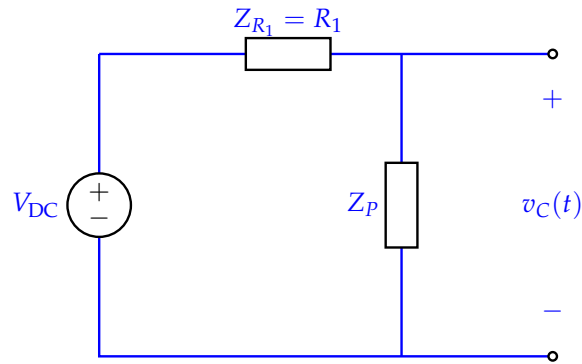


Figure 7

where

$$\frac{1}{Z_P} = \frac{1}{R_2} + \frac{1}{j\omega L} + j\omega C \quad (45)$$

Hence, from the voltage divider formula, we have

$$H(f) = \frac{V_{\text{out}}}{V_{\text{in}}} \quad (46)$$

$$= \frac{Z_P}{Z_P + R_1} \quad (47)$$

$$= \frac{1}{1 + \frac{R_1}{Z_P}} \quad (48)$$

$$= \frac{1}{1 + \frac{R_1}{R_2} + j\omega R_1 C - \frac{jR_1}{\omega L}} \quad (49)$$

$$= \frac{1}{1 + \frac{R_1}{R_2} + jR_1 C \omega_0 \left(\frac{\omega}{\omega_0} - \frac{1}{\omega L C \omega_0} \right)} \quad (50)$$

$$= \frac{1}{1 + \frac{R_1}{R_2} + jR_1 C \omega_0 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)} \quad (51)$$

$$= \frac{1}{\underbrace{1 + \frac{R_1}{R_2}}_{k_1} + j \underbrace{R_1 C \omega_0}_{k_2} \left(\frac{f}{f_0} - \frac{f_0}{f} \right)} \quad (52)$$

$$(53)$$

Plugging in values for R_1 , R_2 , L , C , and ω_0 , we get

$$H(f) = \frac{1}{2 + j2 \left(\frac{2\pi f}{10^9} + \frac{10^9}{2\pi f} \right)} \quad (54)$$

- (b) (5 pts.) Find $|H(f)|$ as $f \rightarrow \infty$ and as $f \rightarrow 0$. Then, find the value(s) of f for which $|H(f)|$ is maximized. Qualitatively sketch the magnitude of the transfer function as a function of frequency (f). You may put your answers in terms of k_1 and k_2 if you wish. When you are qualitatively sketching, you may define your own axes and scaling. The important part is to

note the general shape of the curve – you do not have to denote exact values on your coordinate axes. **In your qualitative plot, make sure to indicate the behavior of $|H(f)|$ as $f \rightarrow 0$ and $f \rightarrow \infty$, and label the point in the graph where $|H(f)|$ is maximized.**

Solution: To start off, we can find the magnitude of the transfer function, i.e.,

$$H(f) = \frac{1}{k_1 + jk_2 \left(\frac{f}{f_0} - \frac{f_0}{f} \right)} \quad (55)$$

$$|H(f)| = \frac{1}{\left| k_1 + jk_2 \left(\frac{f}{f_0} - \frac{f_0}{f} \right) \right|} \quad (56)$$

$$= \frac{1}{\sqrt{k_1^2 + k_2^2 \left(\frac{f^2 - f_0^2}{ff_0} \right)^2}} \quad (57)$$

It should be noted that

$$\lim_{f \rightarrow 0} \left(\frac{f^2 - f_0^2}{ff_0} \right)^2 = \infty \quad (58)$$

and, similarly,

$$\lim_{f \rightarrow \infty} \left(\frac{f^2 - f_0^2}{ff_0} \right)^2 = \infty \quad (59)$$

Hence, $\lim_{f \rightarrow 0} |H(f)| = 0$ and $\lim_{f \rightarrow \infty} |H(f)| = 0$. To find where $|H(f)|$ is maximized, we can find the point where $\left(\frac{f^2 - f_0^2}{ff_0} \right)^2 = 0$. This happens when $f - f_0 = 0$, or equivalently, $f = f_0$. Plugging in $f_0 = \frac{10^9}{2\pi}$ and $k_1 = k_2 = 2$, the magnitude plot (on a log-log scale) would be

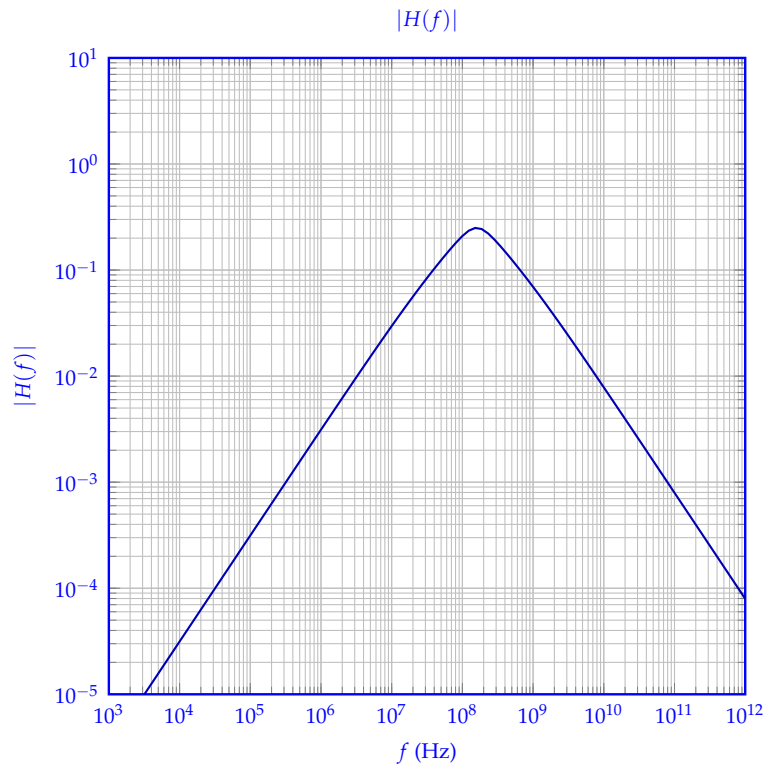


Figure 8: Magnitude Plot Template

- (c) (2 pts.) Using the symbolic transfer function eq. (44), what is the dB magnitude of the transfer function at a frequency of $f = f_0$? Write your answer in terms of k_1 and k_2 , and you do not need to simplify your answer fully.

Solution: Plugging in $f = f_0$ into eq. (44), we get

$$H(f_0) = \frac{1}{k_1 + jk_2(1 - 1)} = \frac{1}{k_1} \quad (60)$$

Hence, the magnitude is

$$|H(f_0)| = \frac{1}{k_1} \quad (61)$$

Recall the the dB magnitude of a transfer function is $20 \log_{10} |H(f)|$, so $20 \log_{10} |H(f_0)| = 20(\log_{10} 1 - \log_{10} k_1) = -20 \log_{10} k_1$.

7. Power is a Complex Topic (15 pts.)

Once again, we are considering the following RLC Circuit with the provided values for R_1, R_2, L, C .

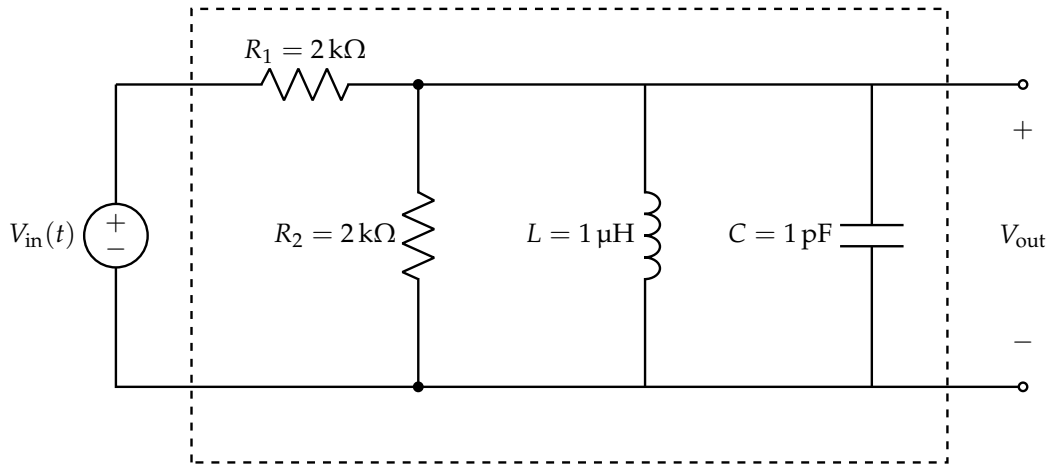


Figure 9

(a) (10 pts.) Now, let's say that the circuit is being driven by a voltage source where

$$V_{in}(t) = 2 \cos(10^9 t + 30^\circ) \quad (62)$$

i. Calculate the real power P .

(HINT: First, find the resonant frequency.)

Solution: Note that the resonant frequency is $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-6}10^{-12}}} = 10^9$. Since $\omega = \omega_0$, the reactive impedances will sum to 0 (see Lecture 8, Slide 10). This means that the total resistance will be the series resistance of the two resistors (since the inductor and capacitor are purely reactive components), so $R_{total} = 4 \times 10^3 \Omega$. Notice also that $V_m = 2$. Thus,

$$P = \frac{V_m^2}{2R_{total}} = \frac{2^2}{2 \times 4 \times 10^3} = 0.5 \times 10^{-3} \text{W} \quad (63)$$

ii. Calculate the reactive power Q .

Solution: From the previous part, we know that there is no reactive impedance, so the reactive power is $Q = 0$.

iii. Calculate the power factor and comment on the relative magnitude of the real and reactive powers.

Solution: From the power triangle concept, we have

$$\cos(\theta) = \frac{P}{\sqrt{P^2 + Q^2}} \quad (64)$$

$$= \frac{P}{P} = 1 \quad (65)$$

so $\theta = 0^\circ$.

(b) (5 pts.) Consider the following circuit in fig. 10. It is known that when $V_{GS} = 1\text{ V}$, $I_{DS} = 1\text{ mA}$ whereas when $V_{GS} = 0\text{ V}$, $I_{DS} = 10^{-9}\text{ A}$.

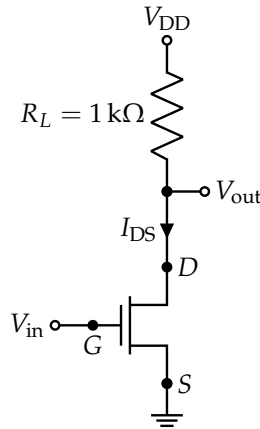
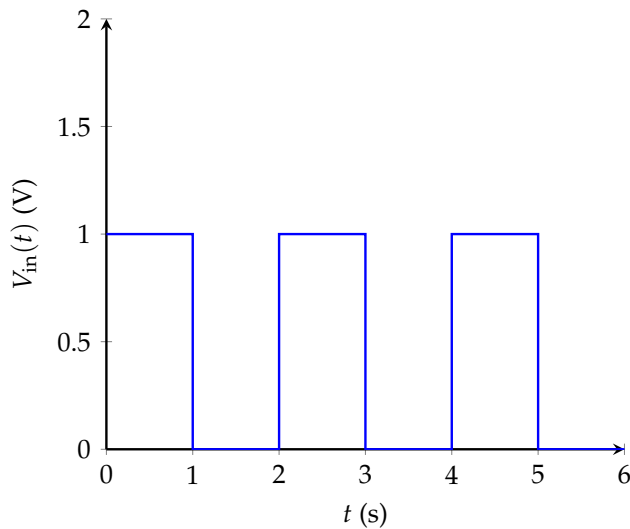
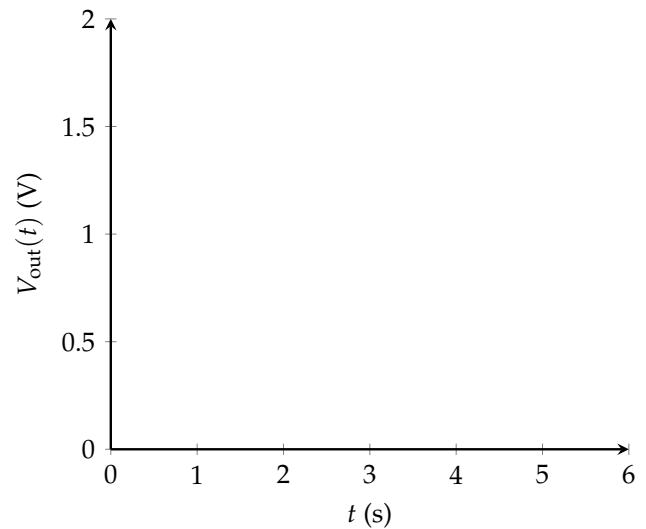


Figure 10: NMOS Transistor Circuit

Given that $V_{DD} = 1\text{ V}$ and the graph for $V_{in}(t)$ is shown in fig. 11a, sketch V_{out} as a function of time t on the plot provided in fig. 11b.



(a) Plot of $V_{in}(t)$



(b) Plot of $V_{out}(t)$

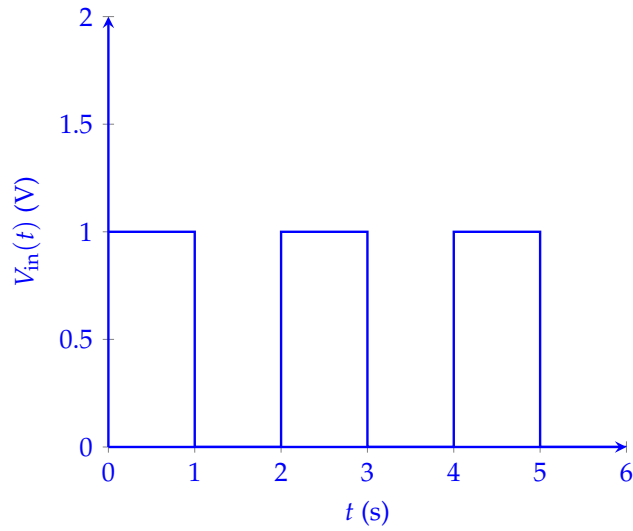
Solution: Notice that $V_{out}(t) = V_{DD} - I_{DS}R_L$ by rearranging Ohm's Law for R_L . We are told that when $V_{GS} = 1\text{ V}$ or when $V_{in} = 1\text{ V}$, the current is $I_{DS} = 1\text{ mA}$. Therefore, when $V_{in} = 1\text{ V}$, we have that

$$V_{out}(t) = 1 - 10^{-3} \times 10^3 = 0\text{ V} \tag{66}$$

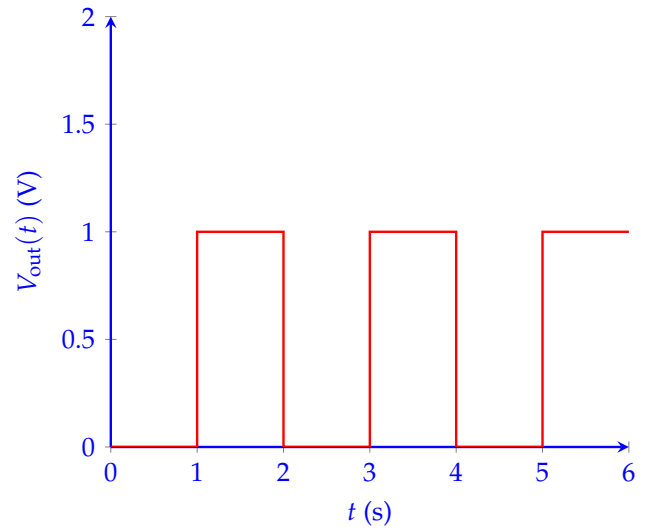
On the contrary, when $V_{GS} = 0\text{ V}$ or when $V_{in} = 0\text{ V}$, the current is $I_{DS} = 10^{-9}\text{ A}$. Therefore, when $V_{in} = 0\text{ V}$, we have"

$$V_{out}(t) = 1 - 10^{-9} \times 10^3 = 1 - 10^{-6}\text{ V} \approx 1\text{ V} \tag{67}$$

Plotting this, we get:



(a) Plot of $V_{in}(t)$



(b) Plot of $V_{out}(t)$

8. Least Squares, Uncertainty, and Safety (30 pts.)

Suppose we have some two dimensional, discrete time system of the form

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_A \vec{x}[i] + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\vec{b}} u[i] \quad (68)$$

where $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ are all unknown, real numbers. For ease of notation, let us denote $\vec{x}[i] := \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}$.

(a) (5 pts.) Let's say we wanted to estimate these parameters using system ID. Upon applying some control inputs, we observe the following data:

i	$x_1[i]$	$x_2[i]$	$u[i]$
0	1	0	2
1	3.41	2.41	1
2	9.95	8.95	3
3	30.97	29.97	7
4	96.86	95.86	3
5	291.71	290.71	3

Set up a least squares problem to solve for the unknowns, i.e., $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$. Then, symbolically describe how to solve the least squares problem. Note: you do not need to actually perform any matrix computations.

Solution: We notice that $a_{11}x_1[i] + a_{12}x_2[i] + b_1u[i] = x_1[i+1]$ and $a_{21}x_1[i] + a_{22}x_2[i] + b_2u[i] = x_2[i+1]$. We know values for $x_1[i], x_2[i], u[i], x_1[i+1], x_2[i+1]$ for $i = 0, 1, \dots, 4$, so we can combine these into a single least squares problem as follows:

$$\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ x_1[2] & x_2[2] & u[2] \\ x_1[3] & x_2[3] & u[3] \\ x_1[4] & x_2[4] & u[4] \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ b_1 & b_2 \end{bmatrix} \approx \begin{bmatrix} x_1[1] & x_2[1] \\ x_1[2] & x_2[2] \\ x_1[3] & x_2[3] \\ x_1[4] & x_2[4] \\ x_1[5] & x_2[5] \end{bmatrix} \quad (69)$$

where, if we were to plug in numbers, we would obtain

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 3.41 & 2.41 & 1 \\ 9.95 & 8.95 & 3 \\ 30.97 & 29.97 & 7 \\ 96.86 & 95.86 & 3 \end{bmatrix}}_S \underbrace{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ b_1 & b_2 \end{bmatrix}}_D \approx \underbrace{\begin{bmatrix} 3.41 & 2.41 \\ 9.95 & 8.95 \\ 30.97 & 29.97 \\ 96.86 & 95.86 \\ 291.71 & 290.71 \end{bmatrix}}_P \quad (70)$$

To solve this least squares problem, we notice that the columns of S are not linearly dependent, so we can write

$$\hat{D} = (S^T S)^{-1} S^T P \quad (71)$$

where \hat{D} is our estimate for D .

- (b) (8 pts.) Regardless of your answer to the previous part, suppose that, when solving the least squares problem, we obtained the following values of A and \vec{b} :

$$A = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} \tag{72}$$

$$\vec{b} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{73}$$

This gives us the following discrete time system:

$$\vec{x}[i + 1] = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} u[i] \tag{74}$$

Diagonalize the system to obtain

$$\vec{x}_\lambda[i + 1] = \Lambda \vec{x}_\lambda[i] + \tilde{\vec{b}}u[i] \tag{75}$$

where Λ is diagonal and $\vec{x}_\lambda[i]$ is the representation of $\vec{x}[i]$ in the eigenbasis of A .

(HINT: You may want to use the fact that the eigenvectors of A are $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.)

Solution: Based on the hint, we can immediately say that $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and by computing

inverses, we have $V^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. As a result, we can say that $\vec{x}_\lambda[i] = V^{-1}\vec{x}[i]$. Fur-

thermore, based on the hint, we can compute the eigenvectors as $A \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \frac{-3}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

$A \left(\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus, we can say that $\Lambda = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$. It remains to find $\tilde{\vec{b}}$, which we find by computing

$$\tilde{\vec{b}} = V^{-1}\vec{b} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{76}$$

- (c) (2 pts.) **Is the system in eq. (74) stable? Explain your answer.**

Solution: No, there is an eigenvalue with magnitude greater than 1 (there is an eigenvalue equal to 3), so the system is not stable.

(d) (15 pts.) Regardless of your answers to the previous parts, suppose that we could write

$$\vec{x}_\lambda[i] = \begin{bmatrix} \lambda_1^i x_{\lambda_1}[0] \\ \lambda_2^i x_{\lambda_2}[0] \end{bmatrix} + \sum_{j=0}^{i-1} \begin{bmatrix} \lambda_1^{i-1-j} u[j] \\ 0 \end{bmatrix} \quad (77)$$

Let's say that, based on our error in our estimate for least squares, we knew that $2 \leq \lambda_1 \leq 4$ and $0 \leq \lambda_2 \leq 2$. Now, suppose that your initial state is $\vec{x}_\lambda[0] = \vec{0}$, and that you want to uniformly bound your control inputs $u[j]$ for $j = 0, \dots, i-1$, so that $|x_{\lambda_1}[i]| + |x_{\lambda_2}[i]| \leq k$. **In other words, find a $\varepsilon > 0$ such that, if $|u[j]| \leq \varepsilon$ for $j = 0, \dots, i-1$, then $|x_{\lambda_1}[i]| + |x_{\lambda_2}[i]| \leq k$.** Your value of ε may depend on i and k .

(HINT: You may use the triangle inequality, i.e., $|\sum_{i=1}^n a_i| \leq \sum_{i=1}^n |a_i|$. You may also use the geometric series formula, i.e., $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$.)

Solution: Since $\vec{x}_\lambda[0] = \vec{0}$, we can write

$$\vec{x}_\lambda[i] = \sum_{j=0}^{i-1} \begin{bmatrix} \lambda_1^{i-1-j} u[j] \\ 0 \end{bmatrix} \quad (78)$$

so $x_{\lambda_1}[i] = \sum_{j=0}^{i-1} \lambda_1^{i-1-j} u[j]$ and $x_{\lambda_2}[i] = 0$. Now, let $|u[j]| \leq \varepsilon$ for $j = 0, \dots, i-1$. Thus, we have that

$$|x_{\lambda_1}[i]| + |x_{\lambda_2}[i]| \leq \sum_{j=0}^{i-1} |\lambda_1^{i-1-j}| |u[j]| \quad (79)$$

$$\leq \varepsilon \sum_{j=0}^{i-1} \lambda_1^{i-1-j} \quad (80)$$

$$= \varepsilon \left(\frac{1 - \lambda_1^i}{1 - \lambda_1} \right) \quad (81)$$

$$\leq \varepsilon \left(\frac{4^i - 1}{3} \right) \quad (82)$$

Since we need this entire term to be $\leq k$, we can choose $\varepsilon = \frac{3k}{4^i - 1}$.