# Final

Exam Location: Draft

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ____________________, ____________________ ____________________
(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): ____________________

Row Number: ____________________ Seat Number: ____________________

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

1. **Honor Code (0 pts.)**

   **Please copy the following statement in the space provided below and sign your name.**

   *As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.*

   **Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.**

   **Solution:** Any attempt to copy the honor code and sign should get full points.

2. **What are you planning to do during your summer break? (2 pts.)**

   **Solution:** Any answer is sufficient.

3. **What's your favorite thing about Berkeley so far? (2 pts.)**

   **Solution:** Any answer is sufficient.

Do not turn this page until the proctor tells you to do so.
You can work on the above problems before time starts.

**4. SVD and the fundamental subspaces (8 pts.)**

Consider a matrix $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$. The compact SVD of A is given by $A = U_r \Sigma_r V_r^\top$ where

$$U_r = \begin{bmatrix} \vec{u}_1 \cdots \vec{u}_r \end{bmatrix} \in \mathbb{R}^{m \times r}, \quad \Sigma_r = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \in \mathbb{R}^{r \times r}, \quad V_r = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_r \end{bmatrix} \in \mathbb{R}^{n \times r}$$

with $\sigma_1 \geq \cdots \geq \sigma_r > 0$ being the singular values of $A$.

(a) (2 pts.) **Which one of the following sets is always guaranteed to form an orthonormal basis for $\text{Col}(A)$?** *(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $\{\vec{u}_1, \cdots, \vec{u}_r\}$

ii. $\{\sigma_1 \vec{u}_1, \ldots, \sigma_r \vec{u}_r\}$

iii. $\{\vec{v}_1, \ldots, \vec{v}_r\}$

iv. $\{\sigma_1 \vec{v}_1, \ldots, \sigma_r \vec{v}_r\}$

| Option | i | ii | iii | iv |
|---|---|---|---|---|
| Answer | ◯ | ◯ | ◯ | ◯ |

**Solution:** Only options i. and ii. form bases for $\text{Col}(A)$. Since the question is asking for an orthonormal basis the correct answer is i.

(b) (2 pts.) **Which one of the following sets is always guaranteed to form an orthonormal basis for $\text{Col}(A^\top)$?** *(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $\{\vec{u}_1, \cdots, \vec{u}_r\}$

ii. $\{\sigma_1 \vec{u}_1, \ldots, \sigma_r \vec{u}_r\}$

iii. $\{\vec{v}_1, \ldots, \vec{v}_r\}$

iv. $\{\sigma_1 \vec{v}_1, \ldots, \sigma_r \vec{v}_r\}$

| Option | i | ii | iii | iv |
|---|---|---|---|---|
| Answer | ◯ | ◯ | ◯ | ◯ |

**Solution:** Only options iii. and iv. form bases for $\text{Col}(A^\top)$. Since the question is asking for an orthonormal basis the correct answer is iii.

Now suppose that the considered $A$ matrix has the following compact SVD components:

$$U_r = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma_r = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(c) (2 pts.) **Using the given compact SVD, state $\alpha$, where $\alpha$ is the tightest upper bound $\|A\vec{x}\| \leq \alpha$ for any $\vec{x}$ such that $\|\vec{x}\| \leq 1$.**

**Solution:** The largest amplification factor of a matrix is given by its largest singular value. Thus for $A$ this is 2.

(d) (2 pts.) **Given the compact SVD, which of the following provides a valid full SVD for $A = U\Sigma V^\top$?** *(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ii. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

iv. $U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

| Option | i | ii | iii | iv |
|---|---|---|---|---|
| Answer | ◯ | ◯ | ◯ | ◯ |

**Solution:** Only option iv. forms a valid SVD as it has orthonormal U and V matrices and has strictly positive singular values ordered from largest to smallest in the $\Sigma$ matrix.

**5. SVD of a matrix with orthogonal columns (4 pts.)**

Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ where $\vec{a}_i^\top \vec{a}_j = 0$ for all $1 \le i, j \le n$ such that $i \neq j$, and $\vec{a}_i^\top \vec{a}_i \neq 0$ for all $i = 1, \ldots, n$. **What is the set of singular values of $A$ *for all such matrices* $A$?**

*(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

(a) $\{0\}$ (all zero)

(b) $\{\sqrt{\|\vec{a}_1\|}, \ldots, \sqrt{\|\vec{a}_n\|}\}$

(c) $\{\|\vec{a}_1\|, \ldots, \|\vec{a}_n\|\}$

(d) $\{\|\vec{a}_1\|^2, \ldots, \|\vec{a}_n\|^2\}$

(e) $\{1\}$ (all one)

| Option | a | b | c | d | e |
|---|---|---|---|---|---|
| Answer | ◯ | ◯ | ◯ | ◯ | ◯ |

**Solution:** (c). The singular values of $A$ are the square roots of the eigenvalues of $A^\top A$. Evaluating this, we have

$$A^\top A = \begin{bmatrix} \vec{a}_1^\top \\ \vdots \\ \vec{a}_n^\top \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} \vec{a}_1^\top \vec{a}_1 & \cdots & \vec{a}_1^\top \vec{a}_n \\ \vdots & \ddots & \vdots \\ \vec{a}_n^\top \vec{a}_1 & \cdots & \vec{a}_n^\top \vec{a}_n \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \|\vec{a}_1\|^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \|\vec{a}_n\|^2 \end{bmatrix} \tag{3}$$

Since this is a diagonal matrix, we can read off its eigenvalues as $\|\vec{a}_1\|^2, \ldots, \|\vec{a}_n\|^2$. Then the singular values of $A$ are $\{\|\vec{a}_1\|, \ldots, \|\vec{a}_n\|\}$ (note that we ask for the *set* of singular values in the question because singular values are specified in order of decreasing size).

**6. Finding the line that closely fits the data (4 pts.)**

Consider the following matrix $A$ that contains three two-dimensional datapoints:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 3 & -1 \end{bmatrix}. \tag{4}$$

The matrix $A$ has two distinct singular values: $\sigma_1 = \sqrt{21}$ and $\sigma_2 = \sqrt{7}$.

Below is a plot of the datapoints in the 2-D plane, where the $x$-axis corresponds to the first entry and the $y$-axis to the second entry of each column. We would like to fit the line $y = \alpha x$ that minimizes the squared sum of perpendicular distances to the datapoints as follows:

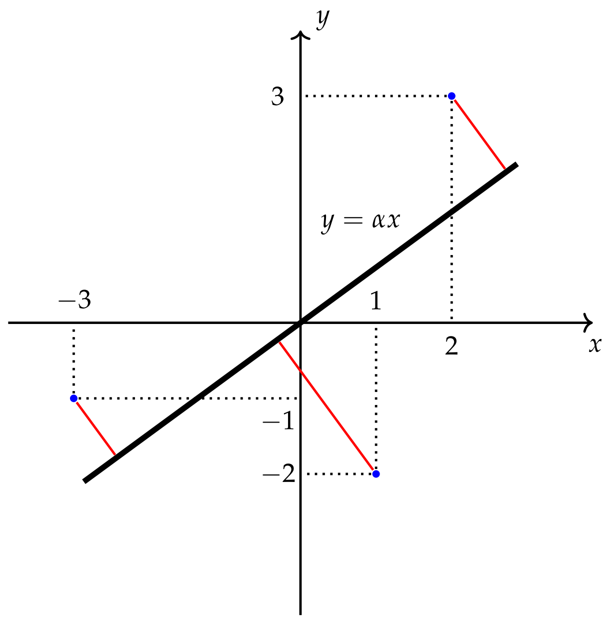



**Figure 1:** Visualization for perpendicular distance minimization.

**Find $\alpha$, using the left singular vectors of matrix $A$. Show your work.**

*(Please fill in one of the circles for the options below.)*

(a) $\alpha = \sqrt{3}/2$

(b) $\alpha = 7/6$

(c) $\alpha = 2$

(d) $\alpha = 1$

| Option | a | b | c | d |
|---|---|---|---|---|
| Answer | ◯ | ◯ | ◯ | ◯ |

**Solution:** The line $y = \alpha x$ that minizes the perpendicular distance will represent the first left singular vector of $A$.

To find the first left singular vector, we can find the eigenvector of $AA^\top$ that corresponds to the largest singular value $\sigma_1 = \sqrt{21}$:

$$AA^\top \vec{u} = \lambda \vec{u} \tag{5}$$

Where $\lambda = \sigma_1^2 = 21$.

Plugging in, we have

$$AA^\top \vec{u} = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ -3 & -1 \end{bmatrix} \vec{u} \tag{6}$$

$$= \begin{bmatrix} 14 & 7 \\ 7 & 14 \end{bmatrix} \vec{u} \tag{7}$$

$$\lambda \vec{u} = 21 \vec{u} \tag{8}$$

Solving for $\vec{u}$ gives $\vec{u} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Checking the slope of $y = \alpha x$, we can conclude that $\alpha = 1$.

| Option | a | b | c | d |
|---|---|---|---|---|
| Answer | ○ | ○ | ○ | ● |

**7. Least squares with repeated columns (4 pts.)**

Consider the following matrix $A$ and vector $\vec{b}$:

$$A = \begin{bmatrix} 0 & 0 & 2 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

We want to find a solution for the following least squares problem:

$$\underset{\vec{x} \in \mathbb{R}^2}{\operatorname{argmin}} \left\| A\vec{x} - \vec{b} \right\|^2 \tag{10}$$

However, we cannot use the least squares solution $\vec{x}_{\text{LS}} = (A^\top A)^{-1} A^\top b$, since $A^\top A$ is not invertible due to the repeated columns in $A$.

We provide a compact SVD of $A$:

$$A = U_r \Sigma_r V_r^\top = \begin{bmatrix} 2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}. \tag{11}$$

**Find a solution for the least squares problem using the Moore-Penrose pseudoinverse.**

**Solution:** A solution for the least squares problem $\operatorname{argmin}_{\vec{x} \in \mathbb{R}^2} \left\| A\vec{x} - \vec{b} \right\|^2$ can be found as following:

$$\begin{aligned}
\vec{x}_* &= A^\dagger b && (12) \\
&= V_r \Sigma_r^{-1} U_r^\top b && (13) \\
&= \begin{bmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} && (14) \\
&= \begin{bmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2/6 & 1/6 & 1/6 \\ 0 & -1/2\sqrt{2} & 1/2\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} && (15) \\
&= \begin{bmatrix} 0 & -1/4 & 1/4 \\ 0 & -1/4 & 1/4 \\ 2/6 & 1/6 & 1/6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} && (16) \\
&= \begin{bmatrix} 1/4 \\ 1/4 \\ 1/2 \end{bmatrix} && (17)
\end{aligned}$$

Note that the solution is also the minimum norm solution.

**8. Matching systems to time responses (4 pts.)**

Consider the following four different 2-D systems $\vec{x}(t)$:

$$\text{System 1:} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \vec{x}(t), \tag{18}$$

$$\text{System 2:} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \vec{x}(t) \tag{19}$$

$$\text{System 3:} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \vec{x}(t) \tag{20}$$

$$\text{System 4:} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \vec{x}(t) \tag{21}$$

where $\vec{x}(t) \in \mathbb{R}^2$ with initial condition $\vec{x}(0) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

We provide the following possible solutions for $\vec{x}(t)$:

(a) $\begin{bmatrix} \frac{15}{4}\mathrm{e}^{t} + \frac{1}{4}\mathrm{e}^{5t} \\ -\frac{5}{4}\mathrm{e}^{t} + \frac{1}{4}\mathrm{e}^{5t} \end{bmatrix}$

(b) $\begin{bmatrix} 4\mathrm{e}^{t} + 3\sqrt{2}t\mathrm{e}^{t} \\ -\mathrm{e}^{t} - 3\sqrt{2}t\mathrm{e}^{t} \end{bmatrix}$

(c) $\begin{bmatrix} 4\mathrm{e}^{3t} - 3\sqrt{2}t\mathrm{e}^{3t} \\ -e^{3t} + 3\sqrt{2}t\mathrm{e}^{3t} \end{bmatrix}$

(d) $\begin{bmatrix} 7\mathrm{e}^{2t} - 3\mathrm{e}^{3t} \\ -7\mathrm{e}^{2t} + 6\mathrm{e}^{3t} \end{bmatrix}$

Each system has one matching solution from the above choices.

**For each system, fill in the circle that matches the correct solution.** *(You will only be graded on your final answer.)*

| Option | a | b | c | d |
|---|---|---|---|---|
| System 1 | ◯ | ◯ | ◯ | ◯ |

| Option | a | b | c | d |
|---|---|---|---|---|
| System 2 | ◯ | ◯ | ◯ | ◯ |

| Option | a | b | c | d |
|---|---|---|---|---|
| System 3 | ◯ | ◯ | ◯ | ◯ |

| Option | a | b | c | d |
|---|---|---|---|---|
| System 4 | ◯ | ◯ | ◯ | ◯ |

**Solution:** Checking the eigenvalues of coefficient matrix of each system, we have:

$$\text{System 1:} \quad \lambda = 1, 5 \tag{22}$$
$$\text{System 2:} \quad \lambda = 3 \tag{23}$$
$$\text{System 3:} \quad \lambda = 2, 3 \tag{24}$$
$$\text{System 4:} \quad \lambda = 1 \tag{25}$$

Matching the corresponding exponential terms in the solution choices, the answer should be as follows.

| Option | a | b | c | d |
|---|---|---|---|---|
| System 1 | ● | ○ | ○ | ○ |

| Option | a | b | c | d |
|---|---|---|---|---|
| System 2 | ○ | ○ | ● | ○ |

| Option | a | b | c | d |
|---|---|---|---|---|
| System 3 | ○ | ○ | ○ | ● |

| Option | a | b | c | d |
|---|---|---|---|---|
| System 4 | ○ | ● | ○ | ○ |

**9. Properties of rotation matrices (8 pts.)**

Consider the $2 \times 2$ matrix

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{26}$$

(a) (2 pts.) **Show that matrix $R$ is orthonormal.**

**Solution:** To show that $R$ is an orthonormal matrix it suffices to show that $R^\top R = RR^\top = I$.

(b) (2 pts.) Consider some vector $\vec{x} \in \mathbb{R}^2$ with norm $\|\vec{x}\|$. **Show that $\|R\vec{x}\| = \|\vec{x}\|$.**

**Solution:** Since $R$ has been shown to be orthonormal, we can conclude that an orthonormal transformation has the property of preserving the norm, i.e. $\|R\vec{x}\| = \|\vec{x}\|$. Alternatively, we can also show this result without knowing this property and simply using the fact that $R^\top R = I$:

$$\|R\vec{x}\|^2 = \vec{x}^\top R^\top R\vec{x} = \vec{x}^\top\vec{x} = \|\vec{x}\|^2 \implies \|R\vec{x}\| = \|\vec{x}\|.$$

(c) (4 pts.) Consider arbitrary real vectors $\vec{a}, \vec{b}$ and let $\beta$ within interval $0 \le \beta \le \pi$ be the angle between them. The inner product for this pair of vectors can be related to angle $\beta$ via the expression $a^\top b = \|\vec{a}\| \left\|\vec{b}\right\| \cos\beta$. Now consider vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ and define $\theta_1$ to be the angle between them. Let $\widetilde{\vec{x}} = R\vec{x}$ and $\widetilde{\vec{y}} = R\vec{y}$ and denote $\theta_2$ to be the angle between $\widetilde{\vec{x}}$ and $\widetilde{\vec{y}}$. **Show that $\theta_1 = \theta_2$, i.e. the angle between two vectors is preserved after an orthonormal transformation.** *(HINT: You may use the inner product expression given at the start of this part as well as results from parts a) and b).)*

**Solution:** From the hint, we know that

$$\begin{aligned} \widetilde{\vec{x}}^\top \widetilde{\vec{y}} &= \left\|\widetilde{\vec{x}}\right\| \left\|\widetilde{\vec{y}}\right\| \cos\theta_2 \\ &= \|\vec{x}\| \, \|\vec{y}\| \cos\theta_2 \end{aligned}$$

where the last line follows from the result of part b).
We also know that

$$\begin{aligned} \widetilde{\vec{x}}^\top \widetilde{\vec{y}} &= (R\vec{x})^\top R\vec{y} \\ &= \vec{x}^\top R^\top R\vec{y} \\ &= \vec{x}^\top \vec{y} \\ &= \|\vec{x}\| \, \|\vec{y}\| \cos\theta_1 \end{aligned}$$

where we used that $R^\top R = I$ due to the orthonormality of $R$.
Putting the results together, we have that

$$\|\vec{x}\| \, \|\vec{y}\| \cos\theta_2 = \|\vec{x}\| \, \|\vec{y}\| \cos\theta_1 \implies \cos\theta_1 = \cos\theta_2.$$

Since the angles in this problem are restricted between 0 and $\pi$, we can conclude that in fact $\theta_1 = \theta_2$.

**10. Inner product with a Hermitian matrix (8 pts.)**

Recall that a matrix $A \in \mathbb{C}^{n\times n}$ is *Hermitian* if $A^* = A$, where $A^*$ denotes the conjugate transpose of $A$. We will show that $A \in \mathbb{C}^{n\times n}$ is Hermitian if and only if for all $\vec{x}, \vec{y} \in \mathbb{C}^n$, $\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A\vec{y}\rangle$.

(a) (3 pts.) Assume that $A \in \mathbb{C}^{n\times n}$ is Hermitian. **Show that for all $\vec{x}, \vec{y} \in \mathbb{C}^n$, $\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A\vec{y}\rangle$.**

**Solution:** We have

$$\begin{aligned} \langle A\vec{x}, \vec{y}\rangle &= \vec{y}^* A\vec{x} && (27)\\ &= \vec{y}^* A^*\vec{x} && (28)\\ &= (A\vec{y})^*\vec{x} && (29)\\ &= \langle \vec{x}, A\vec{y}\rangle && (30) \end{aligned}$$

(b) (5 pts.) Let $A \in \mathbb{C}^{n\times n}$. Assume that for all $\vec{x}, \vec{y} \in \mathbb{C}^n$, $\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A\vec{y}\rangle$. Now we want to show that this implies $A = A^*$. Let $a_{ij}$ and $\widetilde{a}_{ij}$ be the elements of $A$ and $A^*$ respectively. **Pick appropriate vectors $\vec{x}$ and $\vec{y}$ to show that $a_{ij} = \widetilde{a}_{ij}$ for all $1 \le i, j \le n$.**

**Solution:** We have

$$\begin{aligned} \langle A\vec{x}, \vec{y}\rangle &= \vec{y}^* A\vec{x} && (31)\\ \langle \vec{x}, A\vec{y}\rangle &= (A\vec{y})^*\vec{x} && (32)\\ &= \vec{y}^* A^*\vec{x} && (33) \end{aligned}$$

By assumption, $\vec{y}^* A\vec{x} = \vec{y}^* A^*\vec{x}$ are the same for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.
Picking $\vec{x} = e_j$ and $\vec{y} = e_i$ gives us

$$\begin{aligned} \vec{y}^* A\vec{x} &= \vec{y}^* A^*\vec{x} && (34)\\ \implies \vec{e}_i^* A\vec{e}_j &= \vec{e}_i^* A^*\vec{e}_j && (35)\\ \implies a_{ij} &= \widetilde{a}_{ij} && (36) \end{aligned}$$

as desired.

**11. Adaptive cruise control (20 pts.)**

Consider a vehicle traveling with speed $v(t) > 0$ in a road lane behind a "lead" vehicle traveling with constant speed $v_L$. Denote the distance to the lead vehicle by $h(t)$ and the torque input to the follower vehicle with $u(t)$, as shown in Fig. 2:

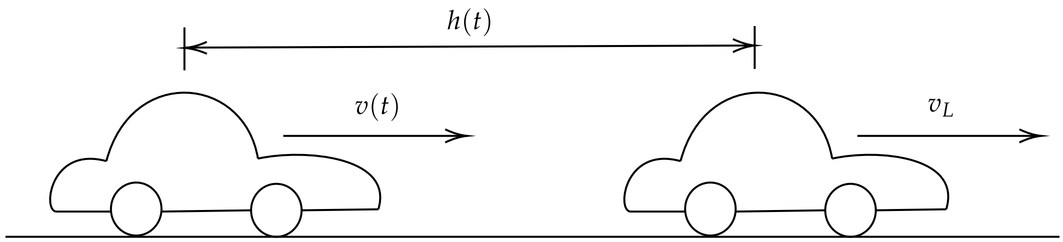



**Figure 2:** Vehicles for adaptive cruise control.

Then a simplified model for $h(t)$ and $v(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t) = v_L - v(t) \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = a - bv(t)^2 + cu(t) \tag{38}$$

where $a$, $b$, $c$ are the appropriate coefficients with $b > 0$ and $c > 0$. We wish to maintain a given relative distance, $h^* > 0$, between the vehicles.

(a) (3 pts.) **Find the values of $v^*$ and $u^*$ that form a valid operating point.**

**Solution:** If $v_*, u_*$ is an operating point, then they satisfy

$$0 = v_L - v_* \tag{39}$$

$$0 = a - bv_*^2 + cu_* \tag{40}$$

From the first equation here, we get that $v_* = v_L$. From the second equation, we get

$$cu_* = -a + bv_*^2 \tag{41}$$

$$u_* = -\frac{a}{c} + \frac{b}{c}v_*^2 \tag{42}$$

$$= -\frac{a}{c} + \frac{b}{c}v_L^2 \tag{43}$$

(b) (5 pts.) The linearization of the system about $(v^*, u^*)$ takes the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\end{bmatrix}\delta u(t) \tag{44}$$

where $\delta h(t) := h(t) - h^*$, $\delta v(t) := v(t) - v^*$, and $\delta u(t) := u(t) - u^*$. **Find the values of $a_{11}$, $a_{12}$, $a_{21}$, $a_{22}$, $b_1$, and $b_2$.**

For reference, the system is

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t) = v_L - v(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = a - bv(t)^2 + cu(t)$$

**Solution:** Let $f_1 := v_L - v(t)$ and $f_2 := a - bv(t)^2 + cu(t)$, so we can write $f := \begin{bmatrix}f_1\\ f_2\end{bmatrix}$. Let $\vec{x} := \begin{bmatrix}h\\ v\end{bmatrix}$. Then

$$J_{\vec{x}}f = \begin{bmatrix}0 & -1\\ 0 & -2bv\end{bmatrix} \tag{45}$$
$$J_u f = \begin{bmatrix}0\\ c\end{bmatrix} \tag{46}$$

so

$$J_x f(\vec{x}_*, u_*) = \begin{bmatrix}0 & -1\\ 0 & -2bv_*\end{bmatrix} \tag{47}$$
$$J_u f(\vec{x}_*, u_*) = \begin{bmatrix}0\\ c\end{bmatrix}. \tag{48}$$

Then

$$\frac{d}{dt}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} = J_{\vec{x}}\vec{f}(\vec{x}_*, u_*)\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} + J_u\vec{f}(\vec{x}_*, u_*)\delta u(t) \tag{49}$$
$$= \begin{bmatrix}0 & -1\\ 0 & -2bv^*\end{bmatrix}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} + \begin{bmatrix}0\\ c\end{bmatrix}\delta u(t) \tag{50}$$

so

$$a_{11} = 0$$
$$a_{12} = -1$$
$$a_{21} = 0$$
$$a_{22} = -2bv^*$$
$$b_1 = 0$$
$$b_2 = c.$$

(c) (12 pts.) Suppose that for some values of $a$, $b$, $c$, and $v_L$, the linearization about some operating point $(v^*, u^*)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} = \begin{bmatrix}0 & -1\\ 0 & -4\end{bmatrix}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\delta u(t) \tag{51}$$

Suppose we apply the feedback law

$$u(t) = u^* + k_1(h(t) - h^*) + k_2(v(t) - v^*), \tag{52}$$

which means

$$\delta u(t) = k_1\,\delta h(t) + k_2\,\delta v(t) \tag{53}$$

**What are the ranges of the feedback gains $k_1$ and $k_2$ that *asymptotically stabilize* this linearized model?**

**Solution:** Applying the feedback law, our system becomes

$$\frac{d}{dt}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} = \begin{bmatrix}0 & -1\\ 0 & -4\end{bmatrix}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\delta u(t) \tag{54}$$

$$= \begin{bmatrix}0 & -1\\ 0 & -4\end{bmatrix}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} \tag{55}$$

$$= \begin{bmatrix}0 & -1\\ k_1 & -4+k_2\end{bmatrix}\begin{bmatrix}\delta h(t)\\ \delta v(t)\end{bmatrix} \tag{56}$$

Let $A := \begin{bmatrix}0 & -1\\ k_1 & -4+k_2\end{bmatrix}$. From the lectures, we know that this $2 \times 2$ system is stable if and only if $\det(A) > 0$ and $\mathrm{Tr}(A) < 0$. This corresponds to

$$\det(A) = k_1 > 0 \tag{57}$$

$$\mathrm{Tr}(A) = -4 + k_2 < 0 \implies k_2 < 4 \tag{58}$$

Alternatively, we can calculate the eigenvalues of the matrix to obtain this result. Let $\alpha_1 := k_1$ and $\alpha_2 := -4 + k_2$ so

$$A = \begin{bmatrix}0 & -1\\ \alpha_1 & \alpha_2\end{bmatrix} \tag{59}$$

Then the eigenvalues of $A$ are given by

$$\lambda_1 = \frac{\alpha_2 - \sqrt{\alpha_2^2 - 4\alpha_1}}{2} \tag{60}$$

$$\lambda_2 = \frac{\alpha_2 + \sqrt{\alpha_2^2 - 4\alpha_1}}{2} \tag{61}$$

If $\alpha_1 \leq 0$ then $\alpha_2 + \sqrt{\alpha_2^2 - 4\alpha_1} \geq \alpha_2 + \sqrt{\alpha_2^2} = \alpha_2 + |\alpha_2| \geq 0$, so $\lambda_2 > 0$.
If $\alpha_2 \geq 0$, we consider when $\alpha_2^2 - 4\alpha_1 < 0$ and when $\alpha_2^2 - 4\alpha_1 \geq 0$. If $\alpha_2^2 - 4\alpha_1 < 0$ then $\sqrt{\alpha_2^2 - 4\alpha_1}$ is imaginary and $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \frac{\alpha_2}{2} \geq 0$. If $\alpha_2^2 - 4\alpha_1 \geq 0$ then $\sqrt{\alpha_2^2 - 4\alpha_1} > 0$ and $\lambda_2 \geq \frac{\alpha_2}{2} \geq 0$.

If $\alpha_1 > 0$ and $\alpha_2 < 0$, we consider when $\alpha_2^2 - 4\alpha_1 < 0$ and when $\alpha_2^2 - 4\alpha_1 \geq 0$. If $\alpha_2^2 - 4\alpha_1 < 0$ then $\sqrt{\alpha_2^2 - 4\alpha_1}$ is imaginary and $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = \frac{\alpha_2}{2} < 0$. If $\alpha_2^2 - 4\alpha_1 \geq 0$ then $0 \leq \alpha_2^2 - 4\alpha_1 < |\alpha_2|$. Then

$$\lambda_1 = \frac{\alpha_2 - \sqrt{\alpha_2^2 - 4\alpha_1}}{2} \leq \frac{\alpha_2}{2} < 0 \tag{62}$$

$$\lambda_2 = \frac{\alpha_2 + \sqrt{\alpha_2^2 - 4\alpha_1}}{2} < \frac{\alpha_2 + |\alpha_2|}{2} \leq 0 \tag{63}$$

Thus we require $\alpha_1 > 0$ and $\alpha_2 < 0$ for the system to be stable. This corresponds to

$$\alpha_1 = k_1 > 0 \tag{64}$$

$$\alpha_2 = -4 + k_2 < 0 \implies k_2 < 4 \tag{65}$$

**12. Minimum-norm input to an RC circuit (20 pts.)**

Consider the following circuit in Fig. 3 used to turn on a light-emitting diode (LED):

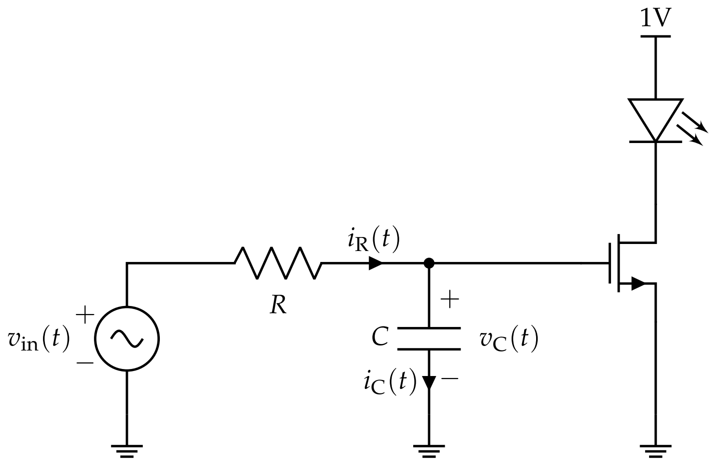



**Figure 3:** Circuit to turn on LED.

The LED turns on when the NMOS gate voltage $v_C(t)$ sufficiently exceeds its threshold voltage. For this problem, let's say we require $v_C(t) \geq 0.5\,\text{V}$ for both the NMOS and the LED to turn on.

The voltage source can generate any continuous-time function $v_{in}(t)$ you desire. The resistor $R$ models the wire resistance, and the capacitor $C$ models the NMOS gate capacitance.

Assume $v_{in}(t) = 0$ for $t < 0$ and $v_C(t) = 0$ for $t < 0$. Your goal is to turn on the LED while minimizing the norm (energy) of the source voltage $v_{in}(t)$.

(a) (3 pts.) **Use KCL to find the continuous-time differential equation for $v_C(t)$ with an arbitrary input voltage $v_{in}(t)$. Write your answer in the form**

$$\frac{\mathrm{d}}{\mathrm{d}t}v_C(t) = Av_C(t) + Bv_{in}(t) \tag{66}$$

**and solve for the coefficients $A$ and $B$.**

**Solution:** With the labeled branch currents on the diagram:

$$i_R = \frac{v_{in}(t) - v_C(t)}{R} \tag{67}$$

$$i_C = C\frac{\mathrm{d}}{\mathrm{d}t}v_C(t) \tag{68}$$

By KCL, we have:

$$\begin{aligned} i_R &= i_C \\ \implies \quad \frac{v_{in}(t) - v_C(t)}{R} &= C\frac{\mathrm{d}}{\mathrm{d}t}v_C(t) \\ \implies \quad \frac{\mathrm{d}}{\mathrm{d}t}v_C(t) &= -\frac{1}{RC}v_C(t) + \frac{1}{RC}v_{in}(t) \end{aligned}$$

and therefore:

$$A = -\frac{1}{RC} \tag{69}$$

$$B = \frac{1}{RC} \tag{70}$$

(b) (5 pts.) Regardless of your previous result, define $x(t) := v_C(t)$ and $u(t) := v_{\text{in}}(t)$ and assume the continuous-time differential equation is of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) - \lambda u(t) \qquad (\lambda < 0) \tag{71}$$

We can discretize this equation assuming a zero-order hold and a uniform sampling period $\Delta$. That is, assuming the discrete forms of $x(t)$ and $u(t)$ are $x_d[i]$ and $u_d[i]$ respectively, where

$$x_d[i] = x(i\Delta) \qquad \text{for } i = 0, 1, 2, \ldots \tag{72}$$

$$u(t) = u_d[i] \qquad \text{for } t \in [i\Delta, (i+1)\Delta) \tag{73}$$

we can write the discrete-time difference equation as

$$x_d[i+1] = A_d x_d[i] + B_d u_d[i] \tag{74}$$

**Solve for $A_d$ and $B_d$ in terms of $\lambda$ and $\Delta$.** Assume that $x_d[0] = 0$.

**Solution:** Recall from Lecture 11 notes, if our continuous-time system is of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \underbrace{\lambda}_{A_c=\lambda} x(t) + \underbrace{b}_{B_c=b} u(t) \tag{75}$$

then this can be discretized as:

$$x_d[i+1] = A_d x_d[i] + B_d u_d[i] \tag{76}$$

where

$$A_d = \mathrm{e}^{\lambda\Delta}, \tag{77}$$

$$B_d = \begin{cases} b\frac{\mathrm{e}^{\lambda\Delta}-1}{\lambda}, & \lambda \neq 0 \\ b\Delta, & \lambda = 0 \end{cases} \tag{78}$$

Therefore, we can map our problem:

$$\frac{\mathrm{d}}{\mathrm{d}x}(t) = \underbrace{\lambda}_{A_c=\lambda} x(t) + \underbrace{(-\lambda)}_{B_c=b=-\lambda} u(t)$$

$$\implies \quad x_d[i+1] = \mathrm{e}^{\lambda\Delta}x_d[i] + b(\frac{\mathrm{e}^{\lambda\Delta}-1}{\lambda})u_d[i]$$

$$\implies \quad x_d[i+1] = \mathrm{e}^{\lambda\Delta}x_d[i] + (-\lambda)(\frac{\mathrm{e}^{\lambda\Delta}-1}{\lambda})u_d[i]$$

$$\implies \quad x_d[i+1] = \mathrm{e}^{\lambda\Delta}x_d[i] + (1-\mathrm{e}^{\lambda\Delta})u_d[i]$$

and therefore by inspection:

$$A_d = \mathrm{e}^{\lambda\Delta} \tag{79}$$

$$B_d = 1 - \mathrm{e}^{\lambda\Delta} \tag{80}$$

(c) (10 pts.) Regardless of your previous result, assume that for the rest of this problem our discrete-time difference equation is:

$$x_d[i+1] = A_d x_d[i] + B_d u_d[i] \tag{81}$$

with

$$A_d = \mathrm{e}^{-0.1} \qquad \text{and} \qquad B_d = 0.1 \tag{82}$$

Let us define our target voltage as $X^\star := 0.5\,\mathrm{V}$. We would like $x_d[i]$ to reach $X^\star$ such that our LED turns on in $\ell$ discrete-time steps using an input sequence $u_d[0], u_d[1], \ldots, u_d[\ell-1]$. We can model this as a control problem with a $1 \times \ell$ controllability matrix at timestep $\ell$ as:

$$C_\ell = \begin{bmatrix} A_d^{(\ell-1)} B_d & \cdots & A_d B_d & B_d \end{bmatrix} \tag{83}$$

such that

$$X^\star = \begin{bmatrix} A_d^{(\ell-1)} B_d & \cdots & A_d B_d & B_d \end{bmatrix} \begin{bmatrix} u_d[0] \\ \vdots \\ u_d[\ell-2] \\ u_d[\ell-1] \end{bmatrix} \tag{84}$$

**Find the minimum-norm sequence for the input $u_d$, i.e. find $u_d[i]$ such that $||\vec{u}_d||^2 = \sum_{i=0}^{\ell-1} |u_d[i]|^2$ is minimized.**

To simplify your arithmetic, use:

$$C_\ell C_\ell^\top = \frac{1}{20}(1 - \mathrm{e}^{-0.2\ell}) \tag{85}$$

**Solution:** If we consider the input sequence as a vector $\vec{u}_d[\ell] = \begin{bmatrix} u_d[i] \end{bmatrix}_{i=0\ldots\ell-1}^\top$, then the minimum-norm input sequence is the solution to:

$$\begin{bmatrix} u_d[0] \\ \vdots \\ u_d[\ell-2] \\ u_d[\ell-1] \end{bmatrix} = C_\ell^\dagger X^\star \tag{86}$$

where $C_\ell^\dagger$ is the pseudo-inverse of the controllability matrix $C_\ell$, i.e. $C_\ell^\dagger = C_\ell^\top (C_\ell C_\ell^\top)^{-1}$. Therefore:

$$\begin{bmatrix} u_d[0] \\ \vdots \\ u_d[\ell-2] \\ u_d[\ell-1] \end{bmatrix} = C_\ell^\top (C_\ell C_\ell^\top)^{-1} X^\star \tag{87}$$

We are given $C_\ell C_\ell^\top = \frac{1}{20}(1 - \mathrm{e}^{-0.2\ell})$, therefore $(C_\ell C_\ell^\top)^{-1} = \frac{20}{1-\mathrm{e}^{-0.2\ell}}$:
We can then solve for our input sequence vector:

$$\begin{bmatrix} u_d[0] \\ \vdots \\ u_d[\ell-2] \\ u_d[\ell-1] \end{bmatrix} = C_\ell^\dagger X^\star \tag{88}$$

$$= C_\ell^\top (C_\ell C_\ell^\top)^{-1} X^\star \tag{89}$$

$$= \begin{bmatrix} c_{\ell-1} \\ \vdots \\ c_1 \\ c_0 \end{bmatrix} \left(\frac{20}{1-e^{-0.2\ell}}\right)(0.5) \tag{90}$$

$$= \frac{10}{1-e^{-0.2\ell}} \begin{bmatrix} 0.1e^{-0.1(\ell-1)} \\ \vdots \\ 0.1e^{-0.1} \\ 0.1 \end{bmatrix} \tag{91}$$

$$= \frac{1}{1-e^{-0.2\ell}} \begin{bmatrix} e^{-0.1(\ell-1)} \\ \vdots \\ e^{-0.1} \\ 1 \end{bmatrix} \tag{92}$$

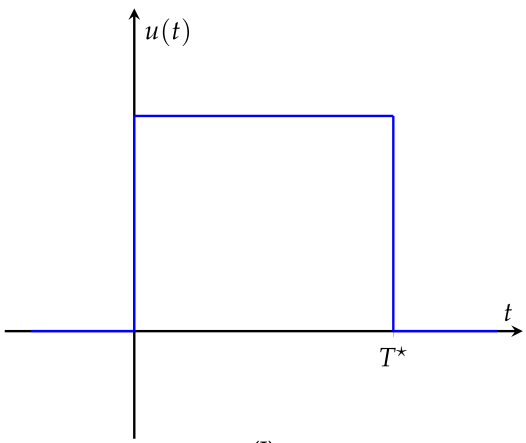
By pattern matching, we can solve for $u_d[i]$ for any arbitrary discrete-time index $i$, as desired:

$$u_d[i] = \frac{1}{1-e^{-0.2\ell}} e^{-0.1(\ell-1-i)} \tag{93}$$

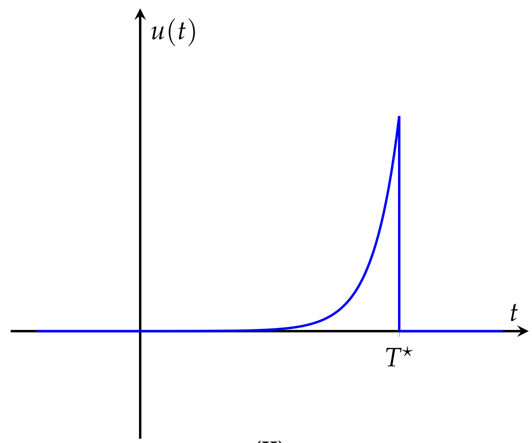
(d) (2 pts.) Assume we want our LED to turn on at target time $t = T^\star$. Beyond that time, the status of the LED is irrelevant, so we turn off our source $u(t)$ for $t > T^\star$.

**In the table below, select the closest continuous-time input source voltage $u(t)$ which corresponds to the minimum-norm solution.** Assume the axes in the plots are all to the same scale. *You will only be graded on your final answer.*

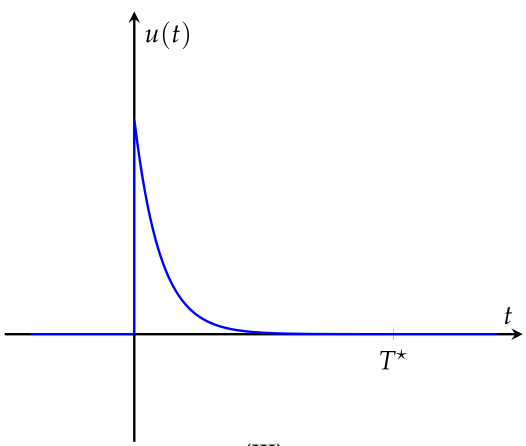
| Option | I | II | III | IV |
|---|---|---|---|---|
| Answer | ◯ | ◯ | ◯ | ◯ |



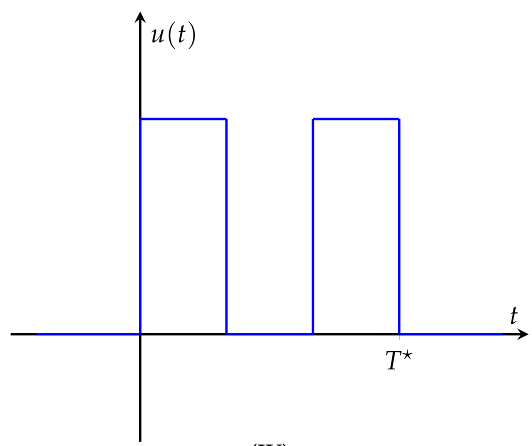

**(I)**




**(II)**




**(III)**




**(IV)**

**Solution:** From the discrete-time solution we found earlier:

$$u_d[i] = \frac{1}{1 - e^{-0.2\ell}} e^{-0.1(\ell-1-i)} \quad (94)$$

$$= \frac{1}{1 - e^{-0.2\ell}} e^{-0.1(\ell-1)} e^{0.1i} \quad (95)$$

we observe qualitatively that this has a positive exponential dependence on the discrete-time index $i$. We would expect this dependence to map directly to the continuous-time variable $t$. Therefore, the continuous-time solution should have a positive exponential dependence as well. Intuitively, the natural decay in the circuit means the earlier input voltages have a less significant impact on our present output voltage, so we can "de-emphasize" the earlier portion and concentrate our input later in time. Therefore, **Plot II** is the correct answer.

| Option | I | II | III | IV |
|---|---|---|---|---|
| Answer | ◯ | • | ◯ | ◯ |

**13. Polyphase electricity (20 pts.)**

Large electric grids such as PG&E generate and distribute not 1 but 3 AC voltages offset by certain phases. This helps efficiently deliver constant power to power-hungry motors in machines of large buildings, such as elevators.

To understand why 3-phase power is used, let's first consider a 1-phase system.

We can model an elevator's motor in a 1-phase system as simply a load resistor $R_M$. The motor is plugged into a common wall outlet which supplies an AC voltage $v(t) = V_S \cos(2\pi f t)$ from the generator through the power transmission lines ("hot" and "neutral"), as shown below. Note that ground is defined at the generator, not at the load.

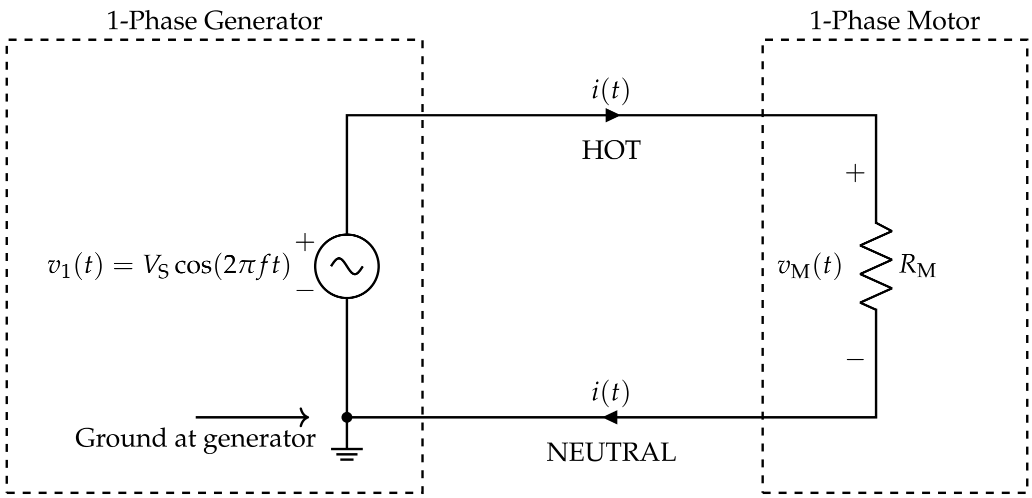



**Figure 5:** 1-phase power.

(a) (1 pts.) What is the current $i(t)$ flowing through the neutral line as a function of time? Write your answer in terms of $V_S$, $R_M$, and $f$.

**Solution:**

$$i(t) = \frac{v(t)}{R_M} = \frac{V_S \cos(2\pi f t)}{R_M} = \frac{V_S}{R_M} \cos(2\pi f t) \tag{96}$$

(b) (2 pts.) Instantaneous electrical power consumed by the motor, $p_M(t)$, is defined as:

$$p_M(t) = v_M(t) i(t) \tag{97}$$

where $v_M(t)$ and $i(t)$ are the voltage and current across the motor (following passive sign convention, as drawn).

The instantaneous power $p_M(t)$ consumed by the 1-phase motor may be written in the form $A + B\cos(2\pi C t)$, where $A$, $B$, and $C$ are real numbers. **Find $A$, $B$, and $C$ in terms of $V_S$, $R_M$, and $f$.**

*(HINT:* $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$*)*

**Solution:**

$$p_M(t) = v_M(t) i_M(t) \tag{98}$$

$$= \frac{v_{\mathrm{M}}^2(t)}{R_{\mathrm{M}}} \tag{99}$$

$$= \frac{(V_{\mathrm{S}}\cos(2\pi 60t)V)^2}{R_{\mathrm{M}}} \tag{100}$$

$$= \frac{V_{\mathrm{S}}^2}{R_{\mathrm{M}}}\left[\frac{1}{2}(1+\cos(2\pi\times 2ft))\right] \tag{101}$$

$$= \frac{V_{\mathrm{S}}^2}{2R_{\mathrm{M}}} + \frac{V_{\mathrm{S}}^2}{2R_{\mathrm{M}}}\cos(2\pi\times 2ft) \tag{102}$$

Therefore:

$$A = \frac{V_{\mathrm{S}}^2}{2R_{\mathrm{M}}} \tag{103}$$

$$B = \frac{V_{\mathrm{S}}^2}{2R_{\mathrm{M}}} \tag{104}$$

$$C = 2f \tag{105}$$

Note that $p_{\mathrm{M}}(t)$ varies across time, i.e. it is not constant.

You should have found that a 1-phase system does not deliver constant power to the elevator motor, resulting in an uneven ride. To fix this, Nikola Tesla was awarded a US patent in 1888 proposing multi-phase power. The simplest case, 2-phase power, is shown below. The voltage sources in a 2-phase generator are phase shifted by 90°, i.e.

$$v_1(t) = V_{\mathrm{S}}\cos(2\pi ft) \tag{106}$$

$$v_2(t) = V_{\mathrm{S}}\cos(2\pi ft - 90^\circ) = V_{\mathrm{S}}\sin(2\pi ft) \tag{107}$$

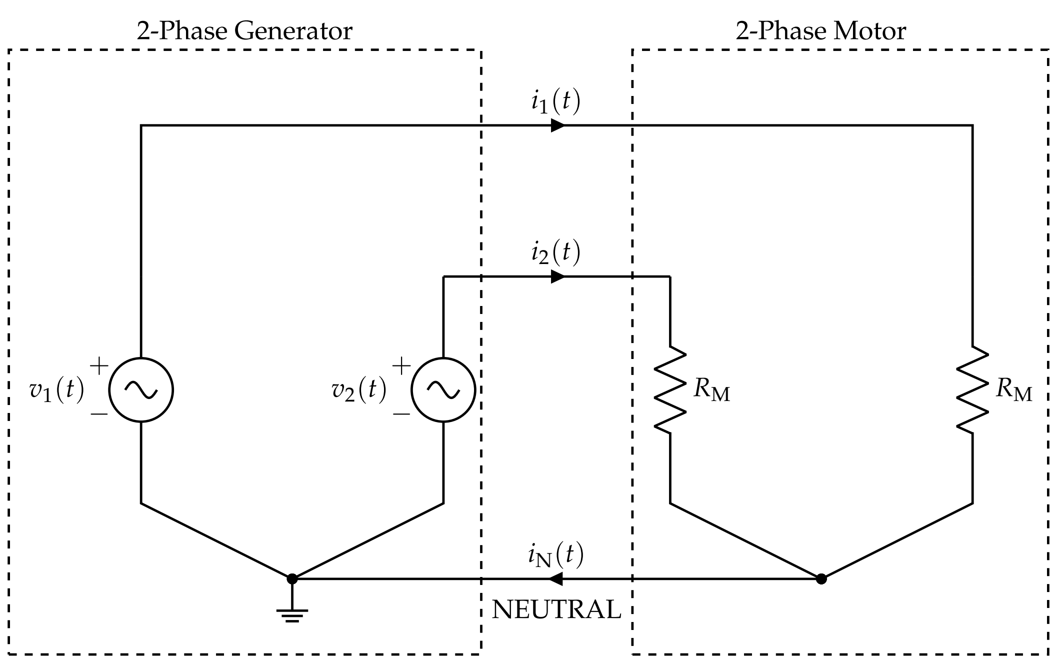



**Figure 6:** 2-phase power.

(c) (5 pts.) The two-phase motor's instantaneous power is defined as $p(t) = p_1(t) + p_2(t)$, where $p_1(t)$ is the power consumed by branch/phase 1 and $p_2(t)$ is the power consumed by branch/phase 2.

**What is the 2-phase motor's total instantaneous consumed power $p(t)$ in terms of $V_S$, $R_M$, and $f$?**

*(HINT:* $\cos^2 x + \sin^2 x = 1$*)* **Solution:** It's easier to do this calculation without directly including the currents:

$$\begin{aligned}
p(t) &= p_1(t) + p_2(t) && (108)\\
&= v_1(t)i_1(t) + v_2(t)i_2(t) && (109)\\
&= v_1(t)\left(\frac{v_1(t)}{R_1}\right) + v_2(t)\left(\frac{v_2(t)}{R_2}\right) && (110)\\
&= \frac{v_1^2(t)}{R_1} + \frac{v_2^2(t)}{R_2} && (111)\\
&= \frac{v_1^2(t)}{R_M} + \frac{v_2^2(t)}{R_M} && (112)\\
&= \frac{V_S^2}{R_M}\left[\cos^2(2\pi f t) + \sin^2(2\pi f t)\right] && (113)\\
&= \frac{V_S^2}{R_M}(1) && (114)\\
&= \frac{V_S^2}{R_M} && (115)
\end{aligned}$$

Note that $p(t)$ is constant over time.

(d) (5 pts.) Now let's use phasors to determine the time-domain neutral current $i_N(t)$ for a 2-phase system.

i. **What are the phasors $\widetilde{V}_1$ and $\widetilde{V}_2$ for the voltage sources $v_1(t)$ and $v_2(t)$, respectively? Write your answers in terms of $V_S$.** **Solution:**

$$\widetilde{V}_1 = \frac{V_S}{2}e^{j0} = \frac{V_S}{2} = \frac{V_S}{2}\angle 0^\circ \quad (116)$$

$$\widetilde{V}_2 = \frac{V_S}{2}e^{j(-90^\circ)} = \frac{V_S}{2}e^{-j\frac{\pi}{2}} = -j\frac{V_S}{2} = \frac{V_S}{2}\angle -90^\circ \quad (117)$$

Any of the above forms is correct.

ii. **What are the phasor $\widetilde{I}_1$ and $\widetilde{I}_2$ for the currents currents $i_1(t)$ and $i_2(t)$, respectively? Write your answers in terms of $V_S$ and $R_M$.** **Solution:**

$$\widetilde{I}_1 = \frac{\widetilde{V}_1}{R_M} = \frac{\frac{V_S}{2}\angle 0^\circ}{R_M} = \frac{V_S}{2R_M}\angle 0^\circ \quad (118)$$

$$\widetilde{I}_2 = \frac{\widetilde{V}_2}{R_M} = \frac{\frac{V_S}{2}\angle -90^\circ}{R_M} = \frac{V_S}{2R_M}\angle -90^\circ \quad (119)$$

Any of the above forms is correct.

iii. The neutral current in a 2-phase system can be written in the form $i_N(t) = A\cos(2\pi Bt + C)$. **Find $A$, $B$, and $C$ in terms of $V_S$, $R_M$, and $f$.** **Solution:** By KCL in the phasor domain:

$$\widetilde{I}_N = \widetilde{I}_1 + \widetilde{I}_2 \quad (120)$$

$$= \frac{V_\mathrm{S}}{2R_\mathrm{M}}\angle 0^\circ + \frac{V_\mathrm{S}}{2R_\mathrm{M}}\angle -90^\circ \tag{121}$$

$$= \frac{\sqrt{2}V_\mathrm{S}}{2R_\mathrm{M}}\angle -45^\circ \tag{122}$$

Converting $\widetilde{I}_\mathrm{N}$ to the time domain:

$$i_\mathrm{N}(t) = \mathrm{Re}\left\{2\widetilde{I}_\mathrm{N}\mathrm{e}^{j2\pi f t}\right\} \tag{123}$$

$$= \mathrm{Re}\left\{2 \times \frac{\sqrt{2}V_\mathrm{S}}{2R_\mathrm{M}}\angle -45^\circ \mathrm{e}^{j2\pi f t}\right\} \tag{124}$$

$$= \frac{\sqrt{2}V_\mathrm{S}}{R_\mathrm{M}}\cos(2\pi f t - 45^\circ) \tag{125}$$

Therefore,

$$A = \frac{\sqrt{2}V_\mathrm{S}}{R_\mathrm{M}} \tag{126}$$

$$B = f \tag{127}$$

$$C = -45^\circ = -\frac{\pi}{4} \tag{128}$$

C may be expressed in degrees or radians as above.

In the early 1900s, electrical engineers at Westinghouse and General Electric proposed a 3-phase distribution method, shown below. One advantage of this 3-phase system is that it delivers 1.5× the constant power compared to a 2-phase system. The 3 generator voltage sources are separated by 120° phase shifts, as defined:

$$v_1(t) = V_S \cos(2\pi f t) \tag{129}$$
$$v_2(t) = V_S \cos(2\pi f t - 120^\circ) \tag{130}$$
$$v_3(t) = V_S \cos(2\pi f t - 240^\circ) \tag{131}$$
$$\tag{132}$$

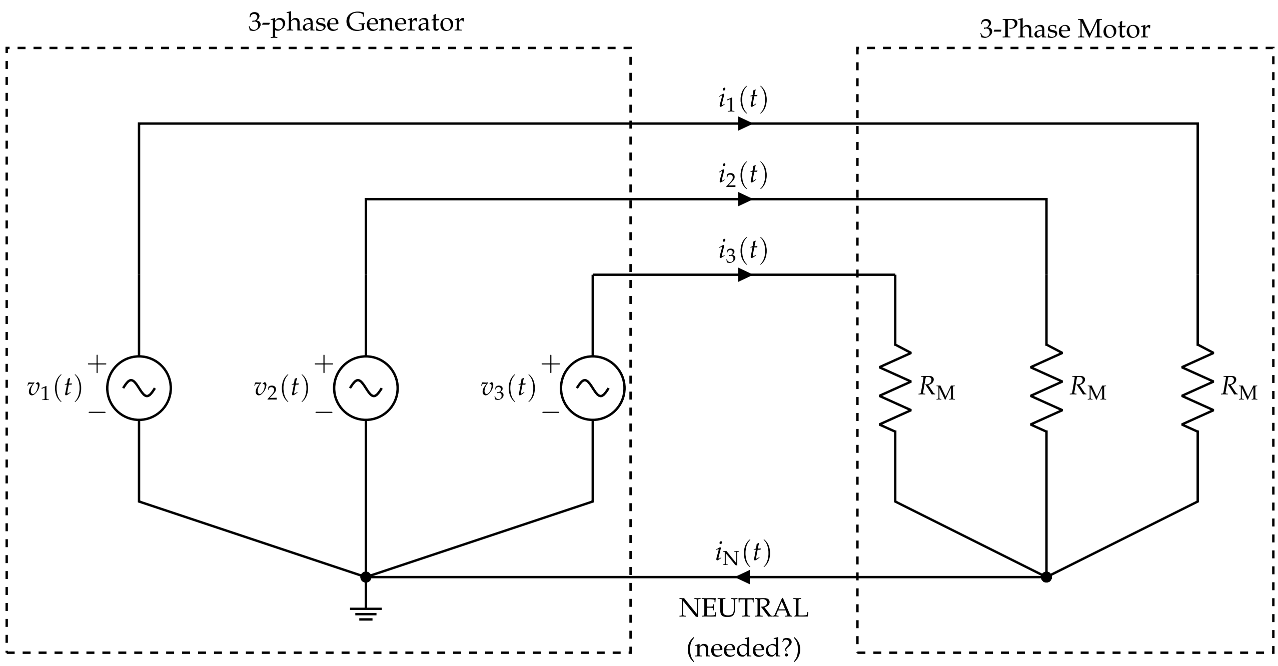



**Figure 7:** 3-phase power.

(e) (7 pts.) In addition to higher constant power delivery, proponents of the 3-phase system claimed the neutral return wire was not required. This potentially reduces the cost connecting an extra transmission line throughout the power grid. **Is this claim true? Justify your answer by finding $i_N(t)$ as a function of time.**

**Solution:** Yes, this is true: $i_N(t) = 0 \quad \forall t$, so there is no need for the neutral return conductor. To show this, we conduct KCL at the neutral node:

$$i_N(t) = i_1(t) + i_2(t) + i_3(t) \tag{133}$$

Analyzing these currents in the phasor domain:

$$\widetilde{I}_N = \widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3 \tag{134}$$
$$= \frac{\widetilde{V}_1}{R_M} + \frac{\widetilde{V}_2}{R_M} + \frac{\widetilde{V}_3}{R_M} \tag{135}$$

$$= \frac{1}{R_M}\left(\frac{V_S}{2}\angle 0^\circ + \frac{V_S}{2}\angle -120^\circ + \frac{V_S}{2}\angle -240^\circ\right) \tag{136}$$

$$= \frac{V_S}{2R_M}\left(1\angle 0^\circ + 1\angle -120^\circ + 1\angle -240^\circ\right) \tag{137}$$

Notice that $1\angle 0^\circ$, $1\angle -120^\circ$, and $1\angle -240^\circ$ are phasors of unit length and rotated around the unit circle symmetrically such that their vector sum is 0. In other words:

$$\widetilde{I}_N = \frac{V_S}{2R_M}(\vec{0}) \tag{138}$$

$$= \vec{0} \tag{139}$$

Therefore:

$$i_N(t) = \text{Re}\left\{2\widetilde{I}_N e^{j2\pi ft}\right\} \tag{140}$$

$$= \text{Re}\{0\} \tag{141}$$

$$= 0 \tag{142}$$

The ability of 3-phase power to deliver constant power to (balanced 3-phase) loads while not requiring a neutral return line is its main advantage over the 2-phase counterpart. This led to the 3-phase system dominating over 2-phase systems in modern power grids worldwide.

**14. Dynamical system approach to solving Ridge Regression (16 pts.)**

In this problem, we will derive a dynamical system based approach to solving a modified version of the least-squares problem, commonly known as "ridge regression". This problem attempts to find the $\vec{x}$ that minimizes $\|A\vec{x} - \vec{y}\|^2 + \lambda \|\vec{x}\|^2$. Here we assume $A \in \mathbb{R}^{m \times n}$ is full column rank and scalar $\lambda \geq 0$.

The solution to the ridge regression problem is

$$\vec{\hat{x}} = (A^\top A + \lambda I)^{-1} A^\top \vec{y}. \tag{143}$$

Note that this solution is quite similar to the solution of least-squares. In many cases, direct computation of the solution to ridge regression is too slow, because it requires computing the matrix inverse $(A^\top A + \lambda I)^{-1}$, which is generally very costly for $A$ with very large dimensions. We will instead solve the problem iteratively by using an update rule which turns this particular problem into an analysis of a particular discrete-time state-space dynamical system.

(a) (2 pts.) We first connect the ridge regression problem to the familiar ordinary least-squares problem. **State the condition on $\lambda$ in (143) needed to recover the least squares solution.**

**Solution:** In ridge regression the solution is $\vec{\hat{x}} = (A^\top A + \lambda I)^{-1} A^\top \vec{y}$. By setting $\lambda = 0$ we recover the formula to compute the solution for least-squares.

(b) (3 pts.) Using (143), **show that $(A^\top A + \lambda I)\vec{\hat{x}} - A^\top \vec{y} = 0$.**

**Solution:** We are given that for ridge regression, the solution is of the form $\vec{\hat{x}} = (A^\top A + \lambda I)^{-1} A^\top \vec{y}$. Plugging this into the left-hand side of the equation that we want to prove, we get

$$(A^\top A + \lambda I)\vec{\hat{x}} - A^\top \vec{y} = (A^\top A + \lambda I)(A^\top A + \lambda I)^{-1} A^\top \vec{y} - A^\top \vec{y} = A^\top \vec{y} - A^\top \vec{y} = 0.$$

(c) (5 pts.) In iterative optimization schemes, we will get a sequence of estimates for $\vec{\hat{x}}$ at each timestep. Let $\vec{x}[i]$ denote our estimate for $\vec{\hat{x}}$ at timestep $i$.

In this problem we will consider the following update rule for solving the ridge regression problem:

$$\vec{x}[i+1] = \vec{x}[i] - \alpha\left((A^\top A + \lambda I)\vec{x}[i] - A^\top \vec{y}\right) \tag{144}$$

that gives us an updated estimate $\vec{x}[i+1]$ using the previous one $\vec{x}[i]$. Here $\alpha$ is the "step size" in our update rule which controls how much we update our solution estimate at each time step. For the purposes of this problem, it doesn't matter where we got the update rule, but the important thing to note is that if $\vec{x}[i] = \vec{\hat{x}}$, then by the previous part, $\vec{x}[i+1] = \vec{\hat{x}}$ and the system remains in equilibrium at $\vec{\hat{x}}$ for all time.

To show that $\vec{x}[i] \to \vec{\hat{x}}$, we define a new state variable $\Delta\vec{x}[i] = \vec{x}[i] - \vec{\hat{x}}$. It represents the deviation from where we want to be.

**Derive the discrete-time state evolution equation for $\Delta\vec{x}[i]$, and show that it takes the form:**

$$\Delta\vec{x}[i+1] = (I - \alpha G)\Delta\vec{x}[i]. \tag{145}$$

**What is $G$?**

**Solution:**

$$\begin{aligned}
\Delta\vec{x}[i+1] &= \vec{x}[i+1] - \vec{\hat{x}} &(146)\\
&= \vec{x}[i] - \alpha\left((A^\top A + \lambda I)\vec{x}[i] - A^\top \vec{y}\right) - \vec{\hat{x}} &(147)\\
&= (\vec{x}[i] - \vec{\hat{x}}) - \alpha\left((A^\top A + \lambda I)\vec{x}[i] - A^\top \vec{y}\right) &(148)\\
&= \Delta\vec{x}[i] - \alpha\left((A^\top A + \lambda I)\vec{x}[i] - A^\top \vec{y}\right) &(149)\\
&= \Delta\vec{x}[i] - \alpha\left((A^\top A + \lambda I)\vec{x}[i] - (A^\top A + \lambda I)\vec{\hat{x}}\right) &(150)\\
&= \Delta\vec{x}[i] - \alpha(A^\top A + \lambda I)\Delta\vec{x}[i] &(151)\\
&= \left(I - \alpha(A^\top A + \lambda I)\right)\Delta\vec{x}[i] &(152)
\end{aligned}$$

So $G = A^\top A + \lambda I$.

(d) (3 pts.) We would like to select $\alpha$ such that $\Delta\vec{x}[i]$ converges to 0. In particular, we want to make sure that we have a stable system. To do this, we need to understand the eigenvalues of $I - \alpha G$. **Given that $\lambda_k\{G\}$ are the eigenvalues of $G$, for $k \in \{1, 2, \ldots, n\}$ what are the eigenvalues of the matrix $I - \alpha G$?** *(Please fill in one of the circles for the options below. You will only be graded on your final answer.)*

i. $1 - \alpha\lambda_k\{G\}$ for $k \in \{1, 2, \ldots, n\}$

ii. $\alpha\lambda_k\{G\}$ for $k \in \{1, 2, \ldots, n\}$

iii. $1 - \lambda_k\{G\}$ for $k \in \{1, 2, \ldots, n\}$

iv. $1 + \alpha\lambda_k\{G\}$ for $k \in \{1, 2, \ldots, n\}$

| Option | i | ii | iii | iv |
|---|---|---|---|---|
| Answer | ◯ | ◯ | ◯ | ◯ |

**Solution:** Suppose that $(\lambda_k\{G\}, \vec{v}_k\{G\})$ is an eigenvalue-eigenvector pair for $G$. Then

$$(I - \alpha G)\vec{v}_k\{G\} = \vec{v}_k\{G\} - \alpha G\vec{v}_k\{G\} = \vec{v}_k\{G\} - \alpha\lambda_k\{G\}\vec{v}_k\{G\} = (1 - \alpha\lambda_k\{G\})\vec{v}_k\{G\}. \quad (153)$$

Hence, the eigenvalues of $I - \alpha G$ are $1 - \alpha\lambda_k\{G\}$.

(e) (3 pts.) For system (145) to be stable, we need all the eigenvalues of $I - \alpha G$ to have magnitudes that are smaller than 1 (since this is a discrete-time system). **State the condition on $\alpha$ that would ensure that system (145) is stable.** You may assume that $\lambda_k\{G\}$ are real and $\lambda_k\{G\} > 0$ for $k \in \{1, 2, \ldots, n\}$.

**Solution:** For stability, we require that $|1 - \alpha\lambda_k\{G\}| < 1$ for $k \in \{1, 2, \ldots, n\}$. This is equivalent to the condition $-1 < 1 - \alpha\lambda_k\{G\} < 1$ for $k \in \{1, 2, \ldots, n\}$. Isolating the $\alpha$ term we obtain the condition $0 < \alpha < \frac{2}{\lambda_k\{G\}}$ for $k \in \{1, 2, \ldots, n\}$.