# EECS 16B Summer 2020 Midterm 2 (Form 1)

## Instructions

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**Justify your answers unless otherwise stated.** A correct result without justification will not receive full credit.

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> *Should I commit academic misconduct during this exam, let me receive a failing grade in EECS 16B or dismissal from the University.*

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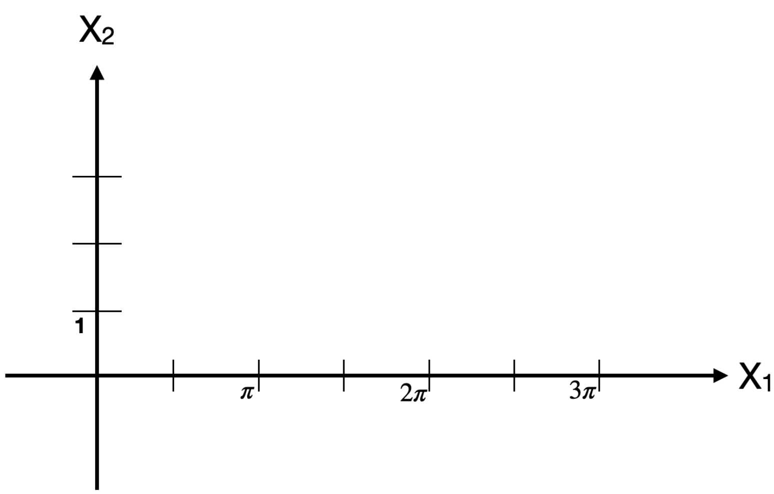
# 1 Equilibria

a) Consider the following non-linear, continuous-time system:

$$\dot{x}(t) = \begin{bmatrix} f_1(x_1(t), x_2(t)) \\ f_2(x_1(t), x_2(t)) \end{bmatrix} = \begin{bmatrix} x_2(t) - sin(x_1(t)) - 1 \\ x_2(t) - 2e^{-x_1(t)} \end{bmatrix}$$

where $x(t) \in \mathbb{R}^2$.

i) Plot the conditions satisfying $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$ on the same plot. Don't worry about finding the exact values for which the two curves intersect, so long as your plot is qualitatively accurate.




**Solution**

ii) How many equilibrium points does this system have? Explain.

**Solution**

This system has infinite equilibria. The decaying exponential will never cross the $x_1$ axis, therefore it will intersect the sinusoid infinitely many times.

figs/p1_plot_sol.png

b) Consider the following non-linear, discrete-time system:

$$x[k+1] = \begin{bmatrix} f_1(x_1[k], x_2[k]) \\ f_2(x_1[k], x_2[k]) \end{bmatrix} = \begin{bmatrix} sin(\frac{\pi}{2} x_1[k]) \\ 1 - x_1[k] + x_2[k]^2 \end{bmatrix}$$

where $x[k] \in \mathbb{R}^2$. What are the equilibria of this system?

### Solution

Candidates for $x_1$ are those values where $x_1 = sin(\frac{\pi}{2} x_1)$, which are given by $x_1 = -1$, $x_1 = 0$ and $x_1 = 1$. For each of these candidates, we evaluate the second condition of equilibrium for each candidate value of $x_1$.

Assuming $x_1 = -1$, we see there are no real-valued solutions to $x_2 = x_2^2 + 2$, so $x_1 = -1$ is thrown out as a candidate.Note here that the state space is defined to be $\mathbb{R}^2$, and therefore imaginary solutions are not considered.

Continuing, assuming $x_1 = 0$, there are again no real-valued solutions to $x_2 = x_2^2 + 1$, so $x_1 = 0$ is also thrown out as a candidate.

Finally, assuming $x_1 = 1$, the solutions to $x_2 = x_2^2$ are $x_2 = 0$ and $x_2 = 1$. Therefore, there are two real-valued equilibrium points for this system, which are $[1, 0]$ and $[1, 1]$.

## 2 Stability

a) Consider the system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

with $x(t) \in \mathbb{R}^2$. Is this system stable? Explain.

### Solution

The exact values in this problem depend on the version of the midterm, but this system is unstable because at least one eigenvalue of the system has a postitive real part.

b) Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

with $x(t) \in \mathbb{R}^2$. Is this system stable? Explain.

### Solution

The eigenvalues of this system are given by

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}.$$

Therefore both eigenvalues have negative real part and this system is stable.

c) Consider the system

$$x_{k+1} = \begin{bmatrix} -0.25 + j & 0 \\ 0 & -0.25 - j \end{bmatrix} x_k + \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

with $x_k \in \mathbb{C}^2$. Is this system stable? Explain.

### Solution

The eigenvalues of this system are complex. Expressing either eigenvalue as $\lambda = \alpha + j\omega$, we have $|\lambda| = \sqrt{\alpha^2 + \omega^2} > 1$, therefore this system is unstable.

## 3 Controllability

a) Consider the system

$$x_{k+1} = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} u_k$$

with $x_k \in \mathbb{R}^3$, and $u_k \in \mathbb{R}$. Is this system controllable? Explain.

### Solution

Let $C = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$. Although values are different for each version of the exam, $C$ can be seen to not be full rank, since $A^2B = 2AB - B$. Therefore this system is uncontrollable.

b) Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

with $x(t) \in \mathbb{R}^4$, and $u(t) \in \mathbb{R}$. Is this system controllable? Explain.

Hint: Can you determine this without constructing the controllability matrix?

### Solution

Depending on the version of the exam, either the first two or last two dimensions of this vector-valued system have time-derivatives which do not depend on the control input or the other dimensions of the system. Therefore those dimensions are uncontrollable, and it follows that the system is uncontrollable.

c) Consider the system

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} u(t)$$

with $x(t) \in \mathbb{R}^{10}$, and $u(t) \in \mathbb{R}$. Is this system controllable? Explain.

Hint: Can you determine this without constructing the controllability matrix?

## Solution

Depending on the form of the exam, either the first dimension, last dimension, or 5th dimension of this vector-valued system has a time-derivative which does not depend on the control input or the other dimensions of the system. Therefore that dimension is uncontrollable, and it follows that the system is uncontrollable.

## 4 SVD

a) How can the SVD of a matrix $A$ be used to find a basis for the null-space of $A$? Justify your answer (1-2 sentences).

**Solution**

If a matrix $A$ has dimension $n \times m$, then the SVD is given by

$$A = U\Sigma V^* = \begin{bmatrix} U1 & U2 \end{bmatrix} \begin{bmatrix} S & 0_{r\times(m-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}.$$

The columns of $V_2$ form a basis of the nullspace of $A$ because if

$$\begin{aligned} 0 &= Ax \\ &= U\Sigma V x \\ &= \Sigma V x \\ &= \begin{bmatrix} SV_1^* x \\ 0 \end{bmatrix} \end{aligned}$$

The third line is derived from the second by left multiplying both sides by $U^{-1}$. We see then that $V_1^* x = 0$. But since $V$ is a unitary matrix, we can express $x = V_1 z_1 + V_2 z_2$, for $r$ dimensional $z_1$ and $m - r$ dimensional $z_2$. Plugging in we have that $V_1^* V_1 z_1 + V_1^* V_2 z_2 = z_1 = 0$. Therefore any $x = V_1 z_1 + V_2 z_2$ satisfying $Ax = 0$ can be expressed simply by $z_1 = 0$, implying $z_2$ is free and therefore any $x$ such that $Ax = 0$ can be expressed as a linear combination of the columns of $V_2$. Since the columns of $V_2$ are orthogonal, we have that these columns form a basis for the null-space of $A$.

Answers which mentioned the facts $\Sigma V_2^* = 0$ and $V_1^* V_2 = 0$ were awarded full credit.

b) How can the SVD of a matrix $A$ be used to find a basis for the column-space of $A$? Justify your answer (1-2 sentences).

**Solution**

If a matrix $A$ has dimension $n \times m$, then the SVD is given by

$$A = U\Sigma V^* = \begin{bmatrix} U1 & U2 \end{bmatrix} \begin{bmatrix} S & 0_{r\times(m-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}.$$

The columns of $U_1$ form a basis of the column space of $A$ because for any $n$ dimensional $y$ such that there exists an $m$ dimensional $x$ with $Ax = y$, we can express $y$ as

$$\begin{aligned} y &= Ax \\ &= U_1 S V_1^* x \\ &= U_1 z \end{aligned}$$

where we defined $z = SV_1^* x$. Note that since $SV_1^*$ is full row-rank, $z$ can take arbitrary values in its $r$ dimensional vector space, and all columns of $U_1$ are necessary to represent $y$ in general. Therefore, a linear combination of the columns of $U_1$ are both necessary and sufficient to represent general $y = Ax$. Therefore $U_1$ forms a basis for the column space of $A$.

Full credit was given without demonstrating the necessity of all columns of $U_1$.

## 5 Optimal Control

Consider a model of a system given by the following:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

Here, $x(t) \in \mathbb{R}^4$ and $u(t) \in \mathbb{R}^2$.

a) Consider a discretized model of this system of the form

$$x[k+1] = A_d x[k] + B_d u[k],$$

where $x[k] := x(Tk)$, $T$ is the discretization sampling period,

$$u(t) = u[k] : t \in [Tk, T(k+1)),$$

and $A_d$ and $B_d$ are given by

$$A_d := \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_d := \begin{bmatrix} 0.125 & 0 \\ 0.5 & 0 \\ 0 & 0.125 \\ 0 & 0.5 \end{bmatrix}.$$

Find the value of $T$ used in this diescretization. Show your work.

### Solution

Consider a system given by

$$\dot{z}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t).$$

Note that this system can describe the evolution of $x_1(t)$ and $x_2(t)$ under influence of control $u_1(t)$, as well as the evolution of $x_3(t)$ and $x_4(t)$ under influence of control $u_2(t)$. In the interval $t \in [kT, (k+1)T]$, we have:

$$\begin{aligned} z_2(t) &= z_2(kT) + \int_{kT}^{t} v_k d\tau \\ &= z_2(kT) + (t - kT)v_k \\ z_1(t) &= z_1(kT) + \int_{kT}^{t} z_2(\tau) d\tau \\ &= z_1(kT) + \int_{kT}^{t} z_2(kT) + (\tau - kT)v_k d\tau \\ &= z_1(kT) + (t - kT)z_2(kT) + \frac{(t - KT)^2}{2} v_k \end{aligned}$$

Therefore

$$z[k+1] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} z[k] + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} v_k$$

Applying this result to our original system, we have:

$$x[k+1] = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix} u[k]$$

Therefore it follows that $T = 0.5$.

b) Is the **discrete-time** system from part (a) controllable? Explain.

### Solution

We can check controllability of each independent subsystem of this system. The controllability matrix for each subsystem is given by

$$\begin{bmatrix} \frac{T^2}{2} & \frac{3T^2}{2} \\ T & T \end{bmatrix}$$

For $T = 0.5$ this matrix is full rank, and therefore each subsystem is controllable. Since both subsystems are independent, the system is controllable.

c) Let $x[0] := \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top$. Find the minimum value $k$ such that the discrete-time system in part (a) can reach the *particular* goal

$$x[k] = x_{goal} := \begin{bmatrix} 0.5 \\ 2 \\ -2 \\ -8 \end{bmatrix}$$

What are the corresponding controls which achieve this goal?

**Solution**

The goal is achieveable in $k = 1$ timestep. The goals for this problem depend on the version of the exam, but the controls which achieve this goal are found by dividing the second and fourth element of $x_{goal}$ by $T$.

d) Let $x[0] := \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top$. We wish to control the system to the configuration $x[3] = \begin{bmatrix} 4 & 0 & 2 & 0 \end{bmatrix}^\top$, such that the quantity

$$\|u[0]\|_2^2 + \|u[1]\|_2^2 + \|u[2]\|_2^2$$

is minimized. Express the solution to this problem in the form

$$\begin{bmatrix} u^*[2] \\ u^*[1] \\ u^*[0] \end{bmatrix} = G \begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \end{bmatrix},$$

for some matrix $G$. When forming $G$, you may express it symbolically in terms of the matrices $A_d$ and $B_d$.

**Solution**

Define $C = \begin{bmatrix} B_d & A_d B_d & A_d^2 B_d \end{bmatrix}$. Because $x[0] = 0$, we have that the optimization problem to solve is

$$\min_{u[0]\in\mathbb{R}^2, u[1]\in\mathbb{R}^2, u[2]\in\mathbb{R}^2} \quad \left\| \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix} \right\|_2$$

$$\text{such that} \quad \begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \end{bmatrix} = C \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix}$$

Note that $C$ is full row-rank because the discrete-time system is controllable. Therefore the solution of this optimization problem is given by

$$\begin{bmatrix} u[2]^* \\ u[1]^* \\ u[0]^* \end{bmatrix} = C^\top (CC^\top)^{-1} \begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$

Therefore $G = C^\top (CC^\top)^{-1}$.

## 6 System Identification

Consider a discrete-time system with unknown dynamics. Assume that starting from $x_0 = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top$ we applied the following controls to the system, and observed the resulting states:

$$u_0 = 1, \quad u_1 = 2, \quad u_2 = 0, \quad u_3 = 1,$$
$$x_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, x_2 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, x_3 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}, x_4 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

a) Set up a least-squares problem to recover $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 1}$ of a discrete-time model of this system

$$x_{k+1} = Ax_k + Bu_k$$

There is no need to solve for the $A$ and $B$ matrices.

### Solution

Define the modeling error of these observations by

$$e = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \begin{bmatrix} x_0^\top & & u_0 & \\ & x_0^\top & & u_0 \\ x_1^\top & & u_1 & \\ & x_1^\top & & u_1 \\ x_2^\top & & u_2 & \\ & x_2^\top & & u_2 \\ x_3^\top & & u_3 & \\ & x_3^\top & & u_3 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ b_1 \\ b_2 \end{bmatrix}$$

Note that $e \in \mathbb{R}^8$. The optimization problem to solve is then

$$\min_{a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2} \|e\|_2$$

Where $e$ is defined as above.

b) Could the estimates of $A$ and $B$ be uniqueley determined from less observations than those given? Explain.

**Solution**

Only if the observation matrix in the least squares problem remains full-rank. Note that

$$\begin{bmatrix} x_0^\top & u_0 \\ x_1^\top & u_1 \\ x_2^\top & u_2 \end{bmatrix}$$

has two identical columns, and therefore matrix is not full rank. It follows that the matrix

$$\begin{bmatrix} x_0^\top & & u_0 & \\ & x_0^\top & & u_0 \\ x_1^\top & & u_1 & \\ & x_1^\top & & u_1 \\ x_2^\top & & u_2 & \\ & x_2^\top & & u_2 \end{bmatrix}$$

is also not full rank. Therefore the least-squares problem does not have a unique solution for this selection of observations. .

## 7 Feedback Control

Consider the system

$$x[k+1] = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k]$$

where $x[k] \in \mathbb{R}^2$ and $u[k] \in \mathbb{R}$.

a) Assume $u[k]$ is chosen at each time $k$ by a feedback controller given by the matrix $K$, i.e.

$$u[k] := -Kx[k]$$

where $K \in \mathbb{R}^{1\times 2}$. Express the closed-loop model of this system in the form

$$x[k+1] = \hat{A}x[k]$$

for some matrix $\hat{A}$ to find.

**Solution**

$\hat{A} = A - BK$.

b) Find the matrix $K$ such that the closed-loop system from part (a) has eigenvalues $\{0.5, -0.5\}$.

**Solution**

The characteristic polynomial of our desired closed loop system is given by $\lambda^2 - 0.25$. By matching the coefficients of the characteristic polynomial of $\hat{A}$ to the coefficients of the desired characteristic polynomial, we obtain the following:
Form 1: $K_1 = 3.75$, $K_2 = 4$.
Form 2: $K_1 = 8.75$, $K_2 = 6$.
Form 3: $K_1 = -3.75$, $K_2 = 0$.