Midterm

Exam Location: AAPB

PRINT your student ID: ______________________________

PRINT AND SIGN your name: ______________, ______________ ______________
(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): ______________

Row Number: ______________ Seat Number: ______________

Name and SID of the person to your left: ______________________________

Name and SID of the person to your right: ______________________________

Name and SID of the person in front of you: ______________________________

Name and SID of the person behind you: ______________________________

1. **Honor Code (0 pts.)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

   Solution: Any attempt to copy the honor code and sign should get full points.

2. **What's something you're proud of having done this last year? (2 pts.)**

   Solution: Any answer is sufficient.

3. **What fall classes or plans are you excited for? (2 pts.)**

   Solution: Any answer is sufficient.

Do not turn this page until the proctor tells you to do so.
You can work on the above problems before time starts.

4. Complex Numbers (7 pts.)

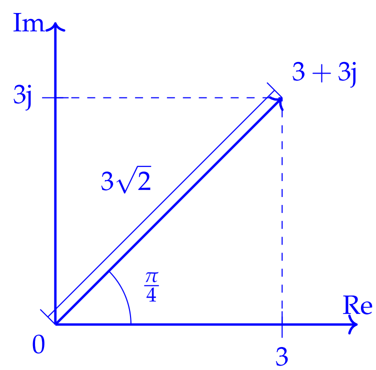
(a) (3 pts.) Let $z_1 = 4e^{j\frac{\pi}{12}}$ and $z_2 = 2e^{j\frac{\pi}{2}}$. **What is** $\left|\frac{z_1}{z_2}\right|$**? What is** $\angle(z_1 \cdot z_2)$**?**

Solution:

- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{|4e^{j\frac{\pi}{12}}|}{|2e^{j\frac{\pi}{2}}|} = \frac{4}{2} = 2$
- $\angle(z_1 \cdot z_2) = \angle(4e^{j\frac{\pi}{12}} \cdot 2e^{j\frac{\pi}{2}}) = \angle(8e^{j\frac{7\pi}{12}}) = \frac{7\pi}{12}$. The angle can also be any value $\frac{7\pi}{12} + 2\pi k$ for an integer k.

(b) (4 pts.) **Convert the voltage phasor** $\widetilde{V}_{\text{out}} = 3 + 3j$ **into a sinusoidal signal** $V_{\text{out}}(t) = A\cos(\omega t + \phi)$**. Specifically, solve for the values of** A **and** ϕ**.**

Solution: The voltage phasor has polar form $\widetilde{V}_{\text{out}} = 3\sqrt{2}e^{j\frac{\pi}{4}}$. $3 + 3j$ is plotted below to depict the angle and magnitude.



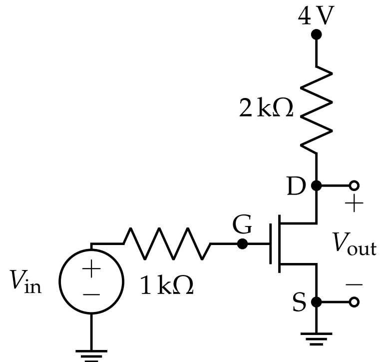
In this class, we use the convention that a phasor is the coefficient of $e^{j\omega t}$. As such, we can write an expression for $V_{\text{out}}(t)$.

$$\begin{aligned}
V_{\text{out}}(t) &= \widetilde{V}_{\text{out}}e^{j\omega t} + \overline{\widetilde{V}_{\text{out}}}e^{-j\omega t} \\
&= 3\sqrt{2}e^{j\frac{\pi}{4}}e^{j\omega t} + \overline{3\sqrt{2}e^{j\frac{\pi}{4}}}e^{-j\omega t} \\
&= 3\sqrt{2}(e^{j(\omega t + \frac{\pi}{4})}) + e^{-j(\omega t + \frac{\pi}{4})}) \\
&= 3\sqrt{2} \cdot 2\cos\left(\omega t + \frac{\pi}{4}\right) \\
&= 6\sqrt{2}\cos\left(\omega t + \frac{\pi}{4}\right)
\end{aligned}$$

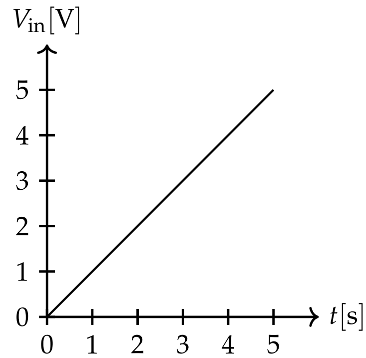
So the sinusoidal voltage V_{out} has amplitude $A = 6\sqrt{2}$, and angle $\phi = \frac{\pi}{4}$.

5. NMOS Logic Inverter (14 pts.)

(a) (14 pts.) We have an NMOS logic implementation of an inverter shown below. The circuit has a voltage input $V_{\text{in}}(t) = t$, $t \geq 0$, ($V_{\text{in}}(t) = 0\,\text{V}$ for $t \leq 0$) seen below.



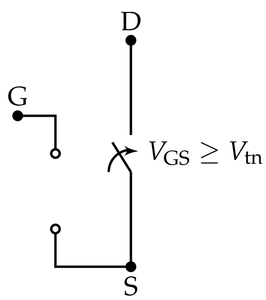
(a) NMOS Inverter



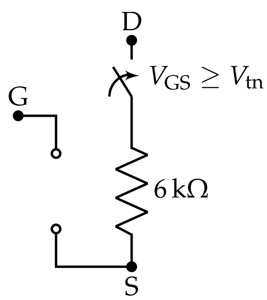
(b) Voltage input, V_{in}

Figure 1: Circuit figure and input signal.

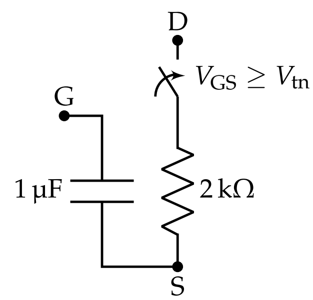
For the transistor models below, define the threshold voltage as $V_{\text{tn}} = 2\,\text{V}$. **Match each NMOS transistor model, plugged into the NMOS inverter circuit, with its corresponding V_{out} plot on the next page.** (Note: All capacitors are fully discharged at $t = 0$.)



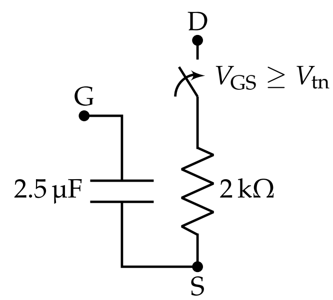
(I) Model I



(II) Model II

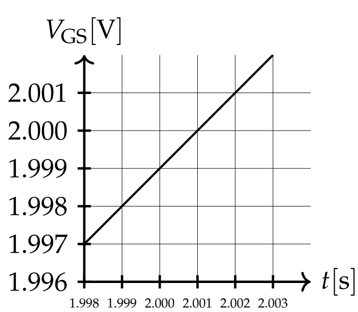


(III) Model III

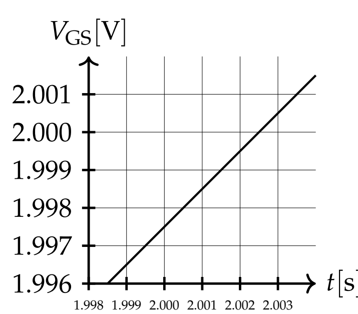


(IV) Model IV

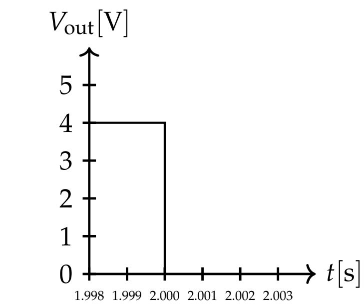
(HINT: You can use the below graphs to evaluate V_{GS} for Models III and IV. We recommend using a scratch page to draw out the NMOS Inverter circuit with the various transistor models plugged in.)



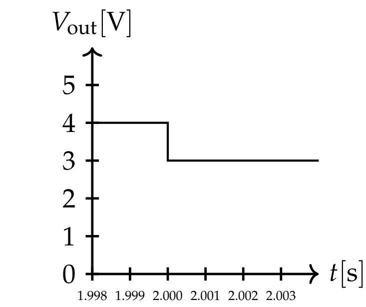
(a) V_{GS} for Model III



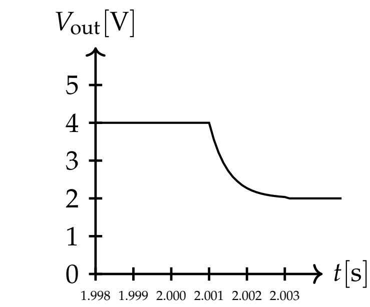
(b) V_{GS} for Model IV



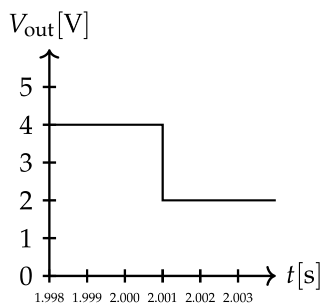
(A) Plot A



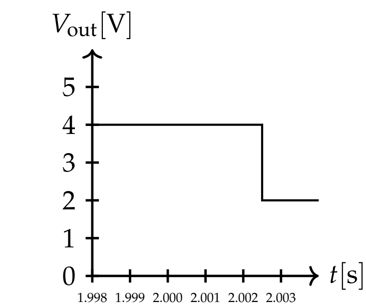
(B) Plot B



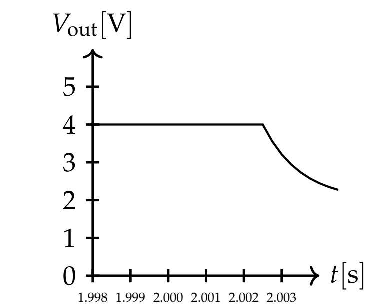
(C) Plot C



(D) Plot D



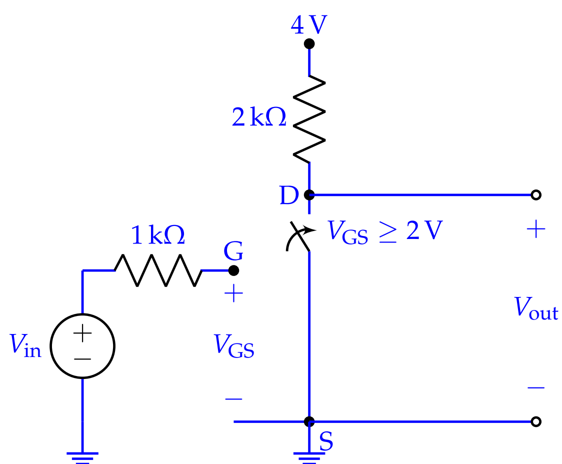
(E) Plot E



(F) Plot F

Model #	Plot A	Plot B	Plot C	Plot D	Plot E	Plot F
I	◯	◯	◯	◯	◯	◯
II	◯	◯	◯	◯	◯	◯
III	◯	◯	◯	◯	◯	◯
IV	◯	◯	◯	◯	◯	◯

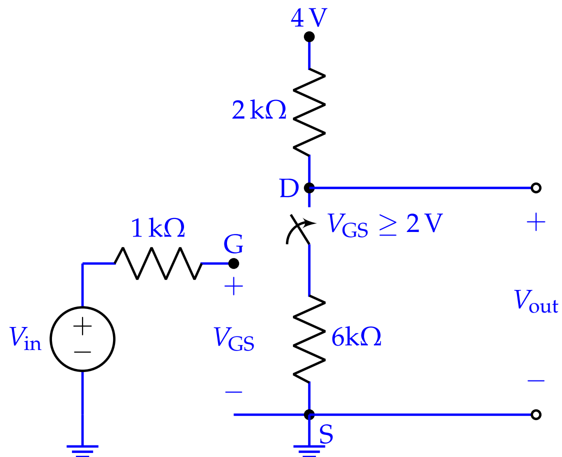
Solution: Let's analyze the four different models, case by case. We start with model I.



First, consider what V_{out} is when the switch is open. No current flows through the upper 2kΩ resistor as it is in series with an open switch. From Ohm's law we have that $\frac{V - V_{out}}{2k\Omega} = 0$, which implies that $V_{out} = 4$V when the switch is open. By similar reasoning, the open over which V_{GS}

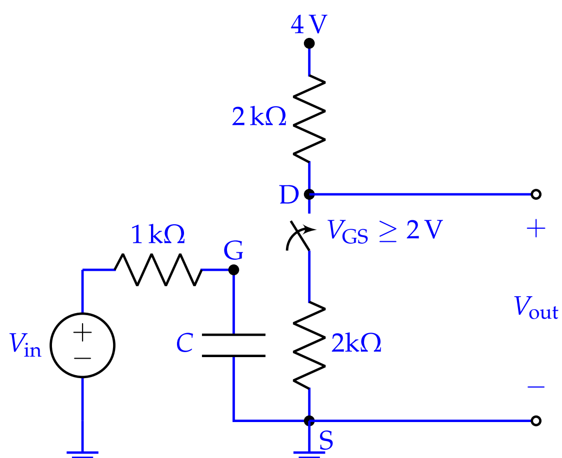
is taken permits no current so $V_{GS} = V_{in}$. If this is the case, we see the transistor hit the threshold voltage, $V_{GS} = 2V$, when V_{in} does at $t = 2s$. So the switch turns on at $t = 2s$, and we should see the voltage on the output instantly drop to ground as there is a short from ground to V_{out}. The plot that matches this behavior of switching from 4V to 0V at $t = 2s$ is plot A.

Let us examine model II. The circuit is depicted below. The only difference is that we now have a 6kΩ resistor from the drain to source, instead of a short of the previous model.



Given that this is the case, we should still see $V_{out} = 4V$ at the beginning. V_{out} should then drop to some voltage at $t = 2s$. However, the voltage that it drops to is now determined by a voltage divider supplied by 4V, with 2kΩ and 6kΩ resistors in series, with the output V_{out} taken over the 6kΩ resistor. This means that $V_{out} = \frac{6k\Omega}{2k\Omega+6k\Omega} \cdot 4V = 3V$ for $t \geq 2s$ when the switch closes. The plot that matches this behavior is plot B.

Consider now models III and IV, for which the only difference is the capacitance value. A circuit with the appropriate model drawn in is shown below.



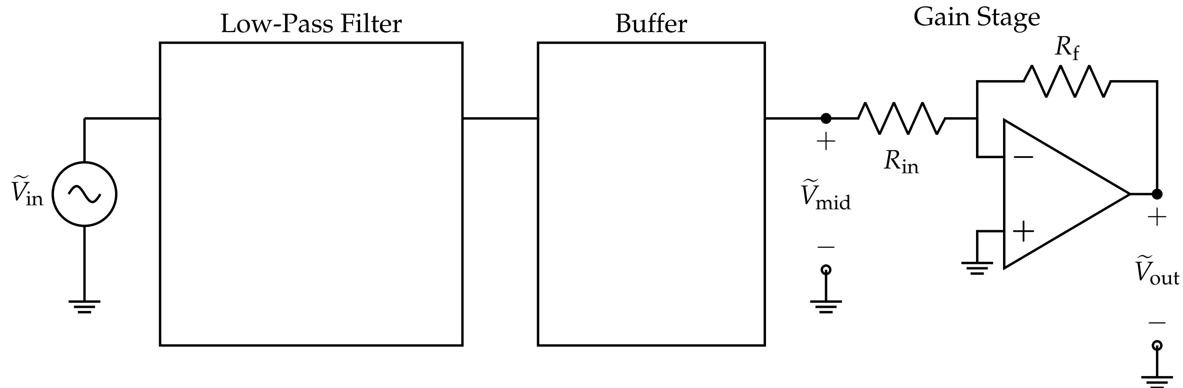
Despite the addition of a capacitor, we still see the output is determined by a voltage divider without any capacitors as in the last case, so we should not see any charging or discharging behavior for V_{out}. This immediately rules out plots C and F, which have exponential decays to some steady state voltage. There is however, charging behavior for V_{GS}, but it is not necessary to solve the differential equation as the switch closure times are indicated by the hint plots as being $t = 2.001$s and $t = 2.0025$s for models III and IV respectively. Intuition regarding RC circuits is an alternative to the hint, as it is the case that it takes a longer time to charge up a larger capacitor, so there is a greater delay for the circuit with $C = 2.5\mu$F. Given that both models have the same output resistor value in the voltage divider of $2k\Omega$, the output voltage should be $V_{\text{out}} = 2$V when the switch closes. Thus for model III, the correct plot is D, and for model IV the correct plot is E. All answers bubbled in are shown in a table below.

Model #	Plot A	Plot B	Plot C	Plot D	Plot E	Plot F
I	●	○	○	○	○	○
II	○	●	○	○	○	○
III	○	○	○	●	○	○
IV	○	○	○	○	●	○

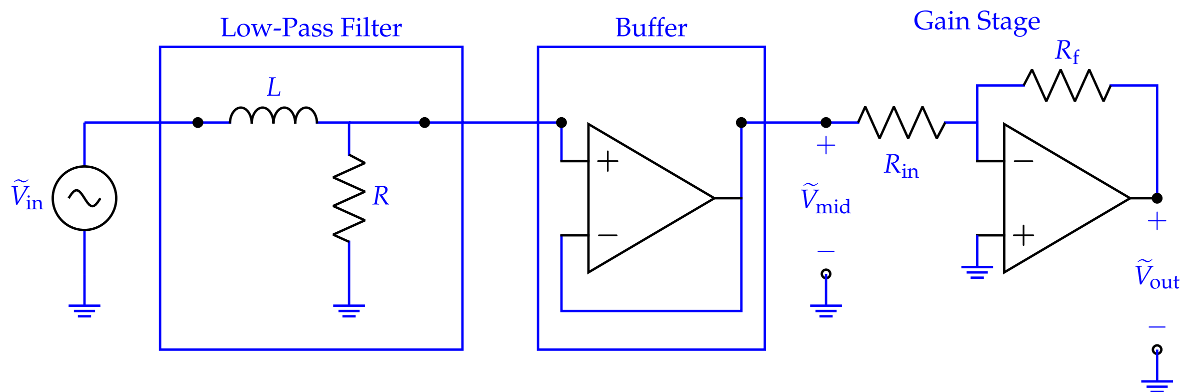
6. Bass Speaker Pre-amplifier (23 pts.)

(a) (4 pts.) Let's design a bass speaker pre-amplifier. We want our pre-amplifier to amplify lower frequencies and attenuate higher frequencies.

In our toolkit, we have **one inductor**, **one op-amp**, and **one resistor** (in addition to the circuit elements already used to implement the Gain Stage in the figure below). Do not worry about the exact values of the inductance and resistance just yet. **Use these components and draw in the circuits that implement the rest of our pre-amplifier in the boxes in the figure below.**



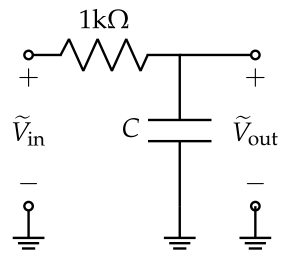
Solution:



The correct ordering of L and R in the low-pass filter can be checked by computing the transfer function $H(\mathrm{j}\omega) = \frac{1}{1+j\omega\frac{L}{R}}$ of the voltage divider formed by L and R. The buffer is an op-amp in negative feedback. Recall that we use the buffer to prevent the loading effect between the different elements in the circuit. It enables us to treat the overall system transfer function as a cascade (mathematically, the multiplicative product) of the transfer functions of its constituent parts.

(b) (2 pts.) Instead of using an inductor, consider the low-pass filter constructed using a resistor and capacitor shown below. **If we want a cutoff frequency of $\omega = 10^3 \frac{\text{rad}}{\text{s}}$ what should the capacitance C be, given that the resistance is $1\,\text{k}\Omega$?**

Solution: An RC low-pass filter has a cutoff frequency of $\omega_c = \frac{1}{RC}$, which can be seen from the transfer function $H(\mathrm{j}\omega) = \frac{1}{1+\mathrm{j}\frac{\omega}{\omega_c}} = \frac{1}{1+\mathrm{j}\omega RC}$. Thus, $C = \frac{1}{\omega_c R} = \frac{1}{10^3\frac{\text{rad}}{\text{s}} \cdot 1\text{k}\Omega} = 1\mu\text{F}$.



(c) (7 pts.) We want to achieve a transfer function magnitude of 10 in the Gain Stage of our pre-amplifier circuit. We will find what R_{f} should be, given resistance $R_{\text{in}} = 20\Omega$.

i. **Solve for the transfer function** $H(\mathrm{j}\omega) = \dfrac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{mid}}}$**.**

ii. **Based on your transfer function from the previous subpart what should R_{f} be to achieve a transfer function magnitude of $|H(\mathrm{j}\omega)| = 10$?**

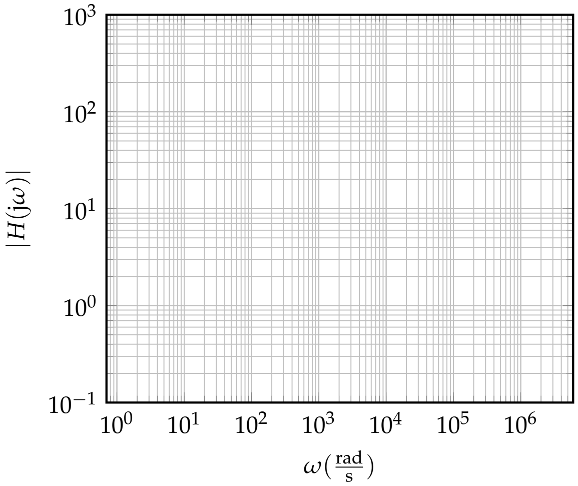
Solution:

i. To solve for the transfer function, if we didn't happen to memorize the inverting amplifier input-output relationship, we can observe that by negative feedback and the positive terminal of the op amp being shorted to ground that the negative terminal must also have the same voltage as ground: 0V. By KCL at the negative input terminal of the op amp we have that $\frac{\widetilde{V}_{\text{mid}} - 0\text{V}}{R_{\text{in}}} = \frac{0\text{V} - \widetilde{V}_{\text{out}}}{R_{\text{f}}}$, which tells us that $H(\mathrm{j}\omega) = \dfrac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{mid}}} = -\frac{R_{\text{f}}}{R_{\text{in}}}$. Either of $-\frac{R_{\text{f}}}{R_{\text{in}}}$ or $-\frac{R_{\text{f}}}{20\Omega}$ are correct.

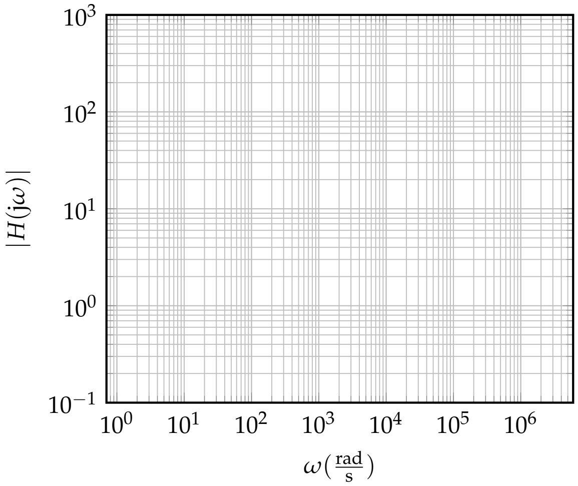
ii. Now, we want that $|H(\mathrm{j}\omega)| = \left|-\frac{R_{\text{f}}}{20\Omega}\right| = \frac{|-R_{\text{f}}|}{|20\Omega|} = \frac{R_{\text{f}}}{20\Omega} = 10$. From the last equality we have that $R_{\text{f}} = 20\Omega \cdot 10 = 200\Omega$.

(d) (6 pts.) We have decided that we will select our inductance and resistance values for the elements from part (a) so that our pre-amplifier can pass all frequencies less than $\omega_{\text{c}} = 10^2 \frac{\text{rad}}{\text{s}}$, and subsequently, amplify all the output by $A_{\text{V}} = 100$.

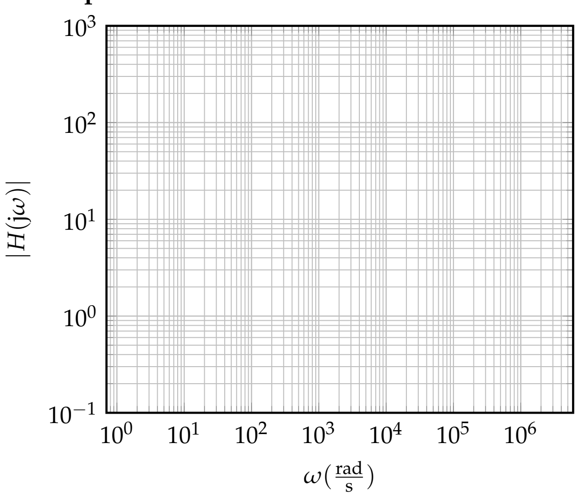
i. **Depict the desired low-pass behavior in a Bode magnitude plot that result from the inductor and the resistor only.**



ii. **Depict the desired Gain Stage behavior in a Bode magnitude plot.**



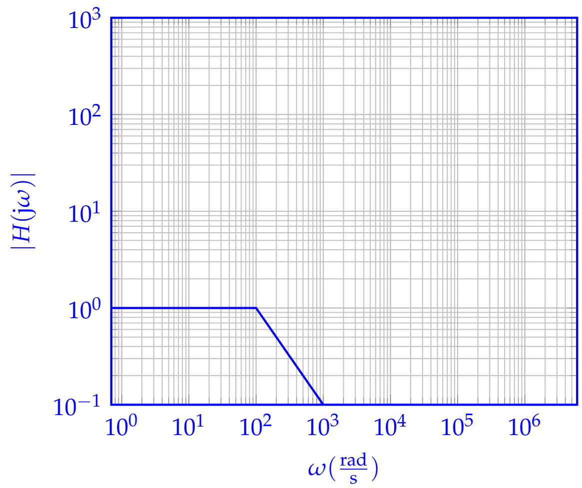
iii. **Depict the desired combined behavior from the previous two plots in a single Bode magnitude plot.**



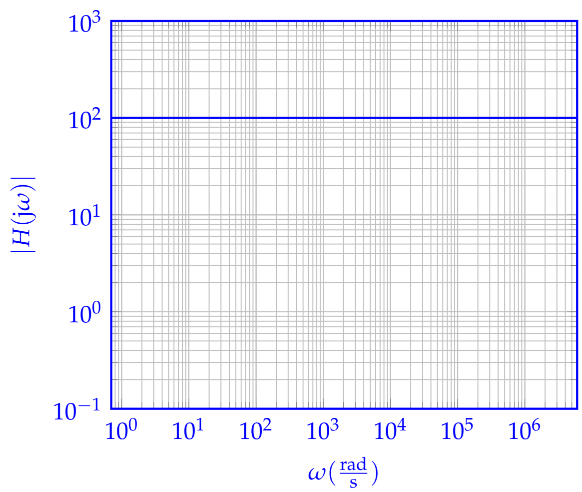
Solution:

i. For the low-pass filter behavior, we have a first order filter with the following straight line approximations in terms of $\log|\omega|$ vs $\log|H(j\omega)|$ relative to $\omega_c = 10^2 \frac{\text{rad}}{\text{s}}$:

- $\omega > \omega_c$: $\log|H_{\text{LPF}}(j\omega)| \approx \log\frac{|\omega_c|}{|\omega|} = \log|\omega_c| - \log|\omega| = 2 - \log|\omega|$.
- $\omega < \omega_c$: $\log|H_{\text{LPF}}(j\omega)| \approx \log 1 = 0$.

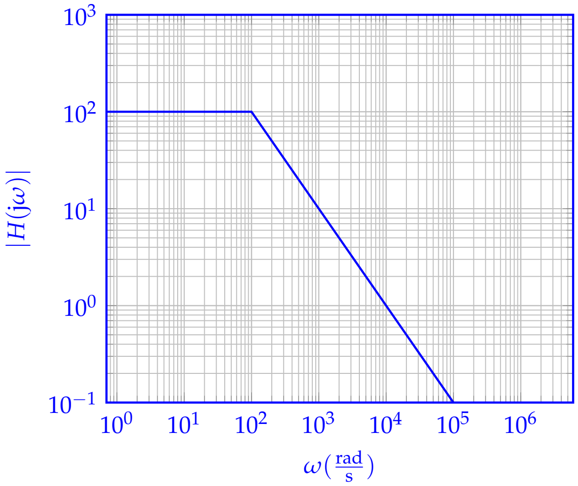


ii. For the Gain Stage behavior, we have a gain of $A_V = 10^2$, which is a constant magnitude of 10^2 (or, a log-magnitude of $\log 10^2 = 2$) for all angular frequencies ω.

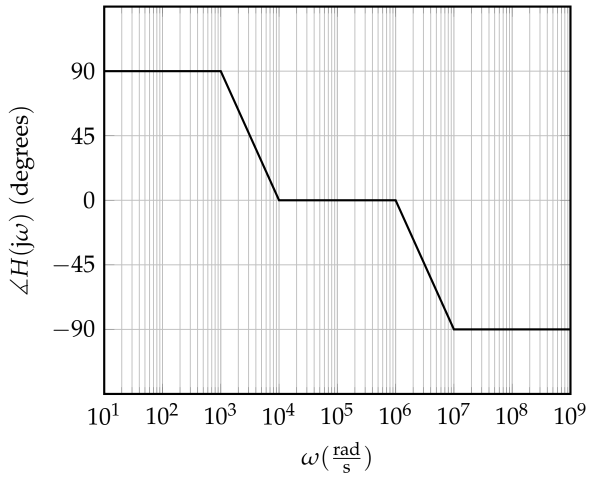


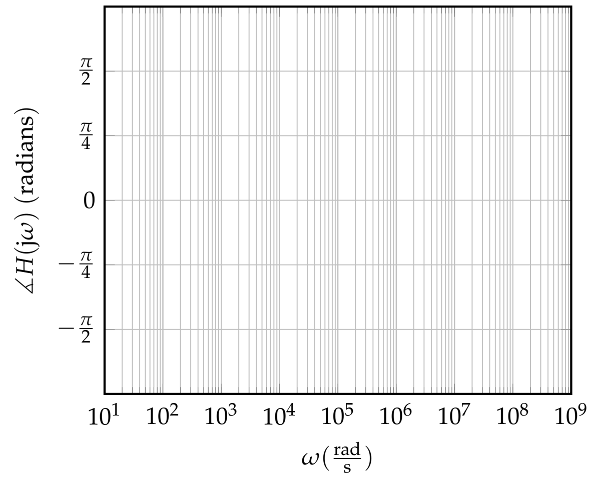
iii. For the combined behavior, we can add the log-magnitudes of the previous stages to get the combined input-output relationship:

- $\omega > \omega_c$: $\log|H_{\text{LPF}}(j\omega)| + 2 \approx 4 - \log|\omega|$.
- $\omega < \omega_c$: $\log|H_{\text{LPF}}(j\omega)| + 2 \approx 2$.


10^3
10^2
10^1
10^0
10^{-1}
$|H(j\omega)|$
10^0 10^1 10^2 10^3 10^4 10^5 10^6
$\omega(\frac{\text{rad}}{\text{s}})$

(e) (4 pts.) Let's say we want to eliminate any phase shift caused by imperfections in the hardware; we'll hence follow the bass speaker pre-amplifier with a hypothetical "phase unshifter" circuit component. **What should the phase plot of the phase unshifter (right plot) look like given the following phase plot (left plot) for the hardware imperfections so that the net phase is 0 for $\omega = 10^1$ to $\omega = 10^9$?** (Note: Both plots share the same y-axis values and scaling.)

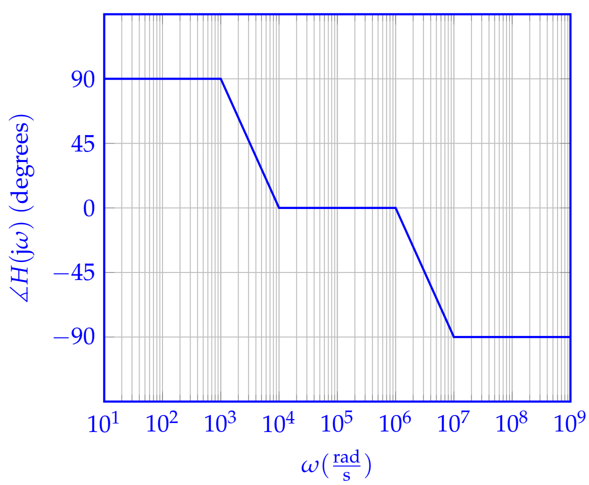




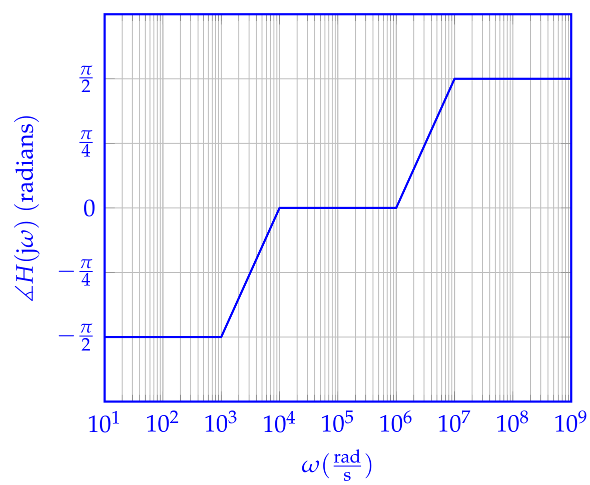
Solution:

When two circuits are connected with loading accounted for, the transfer functions of the circuits multiply. The total transfer function phase will be the sum of each individual circuits transfer function's phase. In particular, for two transfer functions $H_1(j\omega)$ and $H_2(j\omega)$ that represent the input-output relationships of each circuit, the total phase for the input output behavior is $\angle(H_2(j\omega) \cdot H_1(j\omega)) = \angle(H_2(j\omega)) + \angle(H_1(j\omega))$. If we wish for the net phase to be zero, $\angle(H_2(j\omega)) + \angle(H_1(j\omega)) = 0$, which means that we would like our phase-unshifter to have the negative of the angle of the first transfer function: $\angle(H_2(j\omega)) = -\angle(H_1(j\omega))$. The proper plot is depicted below to the right.

Hardware Imperfections

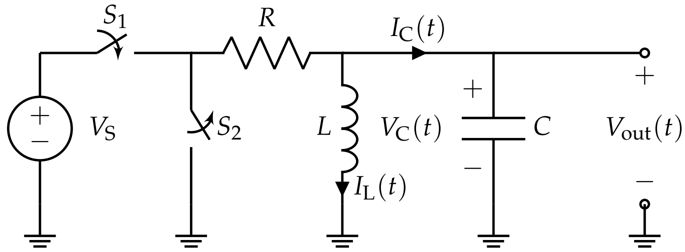


Phase-unshifter



7. RLC Circuit from Time to Frequency (34 pts.)

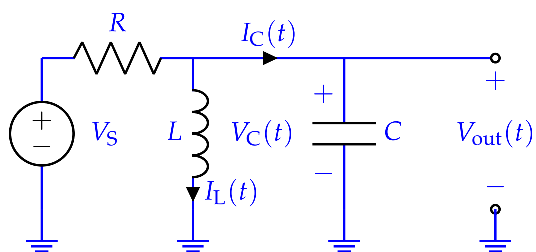
Consider the following circuit fed by a constant voltage source V_S.



The switch S_1, open for $t < 0$, closes at $t = 0$, and the switch S_2, closed for $t < 0$, opens at $t = 0$. Assume $V_C(0) = 0$ and $I_L(0) = 0$.

(a) (8 pts.) **Derive a set of two differential equations, one for $I_L(t)$, the current through the inductor, and one for $V_C(t)$, the voltage across the capacitor.** Write your answer in terms of R, L, C, V_S, and constants.

Solution: The circuit appears as follows:



From Ohm's law for the resistor and KCL at the node with voltage V_C, we have:

$$\frac{V_S - V_C}{R} = I_L + I_C \tag{1}$$

Substituting in the I-V relationship for capacitors in the previous equation, we now have:

$$\frac{V_S - V_C}{R} = I_L + C\frac{dV_C}{dt} \tag{2}$$

$$\frac{V_S}{RC} - \frac{V_C}{RC} = \frac{I_L}{C} + \frac{dV_C}{dt} \tag{3}$$

$$\frac{V_S}{RC} - \frac{V_C}{RC} - \frac{I_L}{C} = \frac{dV_C}{dt} \tag{4}$$

Now, we can notice that $V_C = V_L$ as the inductor and capacitor are in parallel. From the inductor I-V relationship, we have:

$$L\frac{dI_L}{dt} = V_L = V_C \tag{5}$$

$$\frac{dI_L}{dt} = \frac{V_C}{L} \tag{6}$$

In summary, the two differential equations are as follows.

$$\frac{dV_C}{dt} = -\frac{V_C}{RC} - \frac{I_L}{C} + \frac{V_S}{RC} \tag{7}$$

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{V_C}{L} \tag{8}$$

(b) (3 pts.) **Using your answers from the previous part, create a vector differential equation with the state vector being $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$.** Write your answers in terms of R, L, C, V_S, and constants.

Solution: The previous part has the following differential equations.

$$\frac{\mathrm{d}V_C}{\mathrm{d}t} = -\frac{V_C}{RC} - \frac{I_C}{C} + \frac{V_S}{RC} \tag{9}$$

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{V_C}{L} \tag{10}$$

Stacking the above equations into matrix-vector form, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} V_S \tag{11}$$

(c) (15 pts.) Regardless of your answer to the previous part, suppose the vector differential equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \underbrace{\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{\vec{b}} V_S \tag{12}$$

You may use the fact that A is diagonalized as follows:

$$\underbrace{\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -6 & -2 \\ 1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}}_{V^{-1}} \tag{13}$$

With $\vec{x}(0) = \vec{0}$, solve for $\vec{x}(t)$ and find the asymptotic/steady-state behavior as $t \to \infty$.

Solution: We can define $\vec{\widetilde{x}}(t)$ to be the representation of $\vec{x}(t)$ in V-basis. In other words, $\vec{\widetilde{x}}(t) = V^{-1}\vec{x}(t)$ and $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. Applying this, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}V_S \tag{14}$$

$$V^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t)\right) = \underbrace{V^{-1}AV}_{\Lambda}\vec{\widetilde{x}}(t) + V^{-1}\vec{b}V_S \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \Lambda\vec{\widetilde{x}}(t) + \underbrace{\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{V^{-1}\vec{b}} V_S \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \vec{\widetilde{x}}(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} V_S \tag{17}$$

The initial condition is $\vec{\widetilde{x}}(0) = V^{-1}\vec{x}(0) = V^{-1}\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We can solve these equations one row at a time, either by using substitution or the general first order differential equation solution. We use the latter approach. Solve the first row equation for $\widetilde{x}_1(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_1 = -3\widetilde{x}_1 - V_S \tag{18}$$

$$\implies \widetilde{x}_1(t) = -V_S e^{-3t} \int_0^t e^{3\theta}\, d\theta \tag{19}$$

$$= -V_S e^{-3t} \frac{e^{3\theta}}{3} \Big|_{\theta=0}^{\theta=t} \tag{20}$$

$$= -V_S e^{-3t} \frac{e^{3t}-1}{3} \tag{21}$$

$$= V_S \frac{e^{-3t}-1}{3} \tag{22}$$

And now solve the second row equation for $\widetilde{x}_2(t)$.

$$\frac{d}{dt}\widetilde{x}_2 = -\widetilde{x}_2 + V_S \tag{23}$$

$$\implies \widetilde{x}_2(t) = V_S e^{-t} \int_0^t e^{\theta}\, d\theta \tag{24}$$

$$= V_S e^{-t}(e^{t}-1) \tag{25}$$

$$= V_S(1-e^{-t}) \tag{26}$$

so

$$\vec{\widetilde{x}}(t) = V_S \begin{bmatrix} \frac{e^{-3t}-1}{3} \\ 1-e^{-t} \end{bmatrix} \tag{27}$$

Therefore,

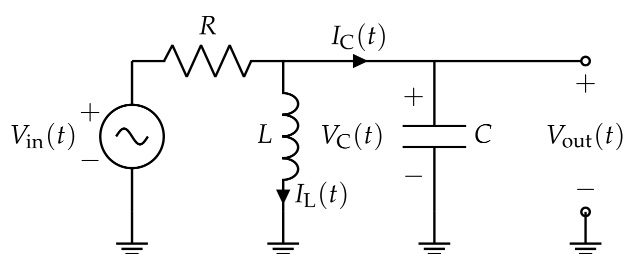
$$\vec{x}(t) = V\vec{\widetilde{x}}(t) = V_S \begin{bmatrix} -6 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{e^{-3t}-1}{3} \\ 1-e^{-t} \end{bmatrix} = V_S \begin{bmatrix} 2\left(e^{-t}-e^{-3t}\right) \\ \frac{e^{-3t}-3e^{-t}+2}{3} \end{bmatrix} \tag{28}$$

Taking a limit as $t \to \infty$, we have

$$\lim_{t\to\infty} \vec{x}(t) = V_S \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} \tag{29}$$

In particular, we can say that $\lim_{t\to\infty} V_C(t) = 0$, and that $\lim_{t\to\infty} I_L(t) = \frac{2}{3}V_S$.

(d) (5 pts.) Now, consider the same circuit but with an arbitrary sinusoidal voltage source instead of the constant voltage source from part (a). Specifically, let $V_{\text{in}}(t) = A\cos(\omega t + \phi)$, for some arbitrary constants A, ω, ϕ. The circuit below reflects this change.



Solve for the transfer function, $H(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$, of this circuit. You may write your answer in terms of R, L, C, ω, and constants.

Solution: The output voltage is taken over the capacitor and the inductor. By treating it as a single impedance we can recognize a voltage divider. Combine the parallel capacitor and inductor impedances as follows:

$$Z_{\text{eq}} = j\omega L || \frac{1}{j\omega C} = \frac{\frac{L}{C}}{j\omega L + \frac{1}{j\omega C}} = \frac{j\omega L}{1-\omega^2 LC} \tag{30}$$

Using the voltage divider formula, we can find $H(\mathrm{j}\omega)$ as follows:

$$\begin{aligned} H(\mathrm{j}\omega) &= \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} && (31)\\ &= \frac{Z_{\text{eq}}}{R + Z_{\text{eq}}} && (32)\\ &= \frac{\frac{\mathrm{j}\omega L}{1-\omega^2 LC}}{R + \frac{\mathrm{j}\omega L}{1-\omega^2 LC}} && (33)\\ &= \frac{\mathrm{j}\omega L}{R + \mathrm{j}\omega L - \omega^2 RLC} && (34) \end{aligned}$$

(e) (3 pts.) **Finally, find $|H(\mathrm{j}\omega)|$ for $V_{\text{in}}(t) = V_S$, where V_S is the voltage supplied by the constant voltage source in part (a).** *(HINT: What would ω be for a constant value?)*

Solution: We can find the magnitude of the transfer function as follows:

$$|H(\mathrm{j}\omega)| = \frac{|\mathrm{j}\omega L|}{|R + \mathrm{j}\omega L - \omega^2 RLC|} = \frac{\omega L}{\sqrt{(R - \omega LC)^2 + (\omega L)^2}} \tag{35}$$

Note that, since V_S is a constant, it can be thought of as the slowest varying sinusoidal signal with an angular frequency of $\omega = 0$ (i.e., there is no oscillation with respect to time). You can also check this by looking at a general sinusoid $f(t) = A\cos(\omega t + \phi)$ and setting $\omega = 0$. We see that $f(t) = A\cos(\phi)$ is a constant with respect to time. Thus, in this case,

$$H(\mathrm{j}0) = \frac{0}{R} = 0 \tag{36}$$

This matches what we had from part 7.c, in that the steady state behavior is captured by phasor domain analysis - any constant voltage fed to the system will lead to an output voltage of zero after a long enough time passes for transients to die out.

8. Straight Line Stability (24 pts.)

We define a discrete and continuous time system respectively as follows:

$$\underbrace{\vec{x}[i+1] = A_d\vec{x}[i] + \vec{b}_d u[i]}_{\text{discrete time}} \qquad \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)}_{\text{continuous time}}$$

where the state is $\vec{x} \in \mathbb{R}^2$, the input to the system is $u \in \mathbb{R}$, and we have parameters $A_d, A \in \mathbb{R}^{2x2}$, and $\vec{b}_d, \vec{b} \in \mathbb{R}^2$.

(a) (8 pts.) **For the following problems, determine whether the system is stable or unstable.**

$$\vec{x}[i+1] = A_d\vec{x}[i] + \vec{b}_d u[i]$$

A_d	**Stable**	**Unstable**
$\begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix}$	◯	◯
$\begin{bmatrix} 0 & 0.25 \\ 0.5 & 0 \end{bmatrix}$	◯	◯

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$$

A	**Stable**	**Unstable**
$\begin{bmatrix} 0 & 7 \\ 0 & -10 \end{bmatrix}$	◯	◯
$\begin{bmatrix} -0.25j & 0 \\ 0 & 0.25j \end{bmatrix}$	◯	◯

Solution:

For $\vec{x}[i+1] = A_d\vec{x}[i] + \vec{b}_d u[i]$, we must check if the magnitude of the eigenvalues of A_d are less than 1. (Note: It's a strict inequality since we don't know the value of $\vec{b}_d$. If $\vec{b}_d = \vec{0}$ and A_d has eigenvalues less than or equal to 1, we could have initial conditions that leads to an $\vec{x}[i]$ that does not decay but does not blow up either.)

- The first matrix $A_d = \begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix}$ is diagonal, so we can read off its diagonal entries for the eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -0.5$. Since $|\lambda_1| = 1 \geq 1$, our system is unstable.
- The second matrix $A_d = \begin{bmatrix} 0 & 0.25 \\ 0.5 & 0 \end{bmatrix}$ is not diagonal, so we must solve for its eigenvalues.

$$0 = \det(A - \lambda I) \tag{37}$$

$$0 = \det\left(\begin{bmatrix} 0 & 0.25 \\ 0.5 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \tag{38}$$

$$0 = \det\left(\begin{bmatrix} -\lambda & 0.25 \\ 0.5 & -\lambda \end{bmatrix}\right) \tag{39}$$

$$0 = \lambda^2 - 0.25(0.5) \tag{40}$$

$$\lambda^2 = 0.25(0.5) \tag{41}$$

$$\lambda = \pm\frac{1}{2\sqrt{2}} \approx \pm 0.353 \tag{42}$$

Since $|\lambda| < 1$, the system is stable.

The correct answers for the discrete-time table are shown below:

$\mathbf{A_d}$	**Stable**	**Unstable**
$\begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix}$	○	●
$\begin{bmatrix} 0 & 0.25 \\ 0.5 & 0 \end{bmatrix}$	●	○

For $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$, we must check if the real portion of the eigenvalues of A are less than 0. (Note: It's a strict inequality since we don't know the value of $\vec{b}$, for a similar reason as in the discrete case.)

- The first matrix $A = \begin{bmatrix} 0 & 7 \\ 0 & -10 \end{bmatrix}$ is upper triangular, so we can read off the diagonal to get the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -10$. Since $Re\{\lambda_1\} = 0 \geq 0$, the system is unstable.
- The second matrix $A = \begin{bmatrix} -0.25j & 0 \\ 0 & 0.25j \end{bmatrix}$ is a diagonal matrix, so we can read off the diagonal to get the eigenvalues $\lambda = 0 \pm 0.25$j. Since $Re\{\lambda\} = 0 \geq 0$, the system is also unstable.

The correct answers for the continuous-time table are shown below:

A	**Stable**	**Unstable**
$\begin{bmatrix} 0 & 7 \\ 0 & -10 \end{bmatrix}$	○	●
$\begin{bmatrix} -0.25j & 0 \\ 0 & 0.25j \end{bmatrix}$	○	●

(b) (8 pts.) Assume that we are operating on a discrete time model ($\vec{x}[i+1] = A_d\vec{x}[i] + \vec{b}_d u[i]$), and our control matrices A_d and $\vec{b}_d$ are fixed as follows:

$$A_d = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, \vec{b}_d = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Let $u[i] = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[i]$. **Solve for the characteristic polynomial of our new feedback-controlled system in the form $C\lambda^2 + D\lambda + E$. You may leave your answer in terms of k_1 and k_2.**

Solution: To solve for the eigenvalue polynomial, let's first solve our discrete time model equation with our given input $u[i]$.

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i] \tag{43}$$

$$= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[i] \tag{44}$$

$$= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 & 0 \\ -k_1 & -k_2 \end{bmatrix} \vec{x}[i] \tag{45}$$

$$= \begin{bmatrix} 3 & -1 \\ 2-k_1 & 1-k_2 \end{bmatrix} \vec{x}[i] \tag{46}$$

We can now solve for the characteristic polynomial of this new matrix.

$$0 = \det(A - \lambda I) \tag{47}$$

$$= \det\left(\begin{bmatrix} 3 & -1 \\ 2-k_1 & 1-k_2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \tag{48}$$

$$= \det\left(\begin{bmatrix} 3-\lambda & -1 \\ 2-k_1 & 1-k_2-\lambda \end{bmatrix}\right) \tag{49}$$

$$= (1-k_2-\lambda)(3-\lambda) + 2 - k_1 \tag{50}$$

$$= 3 - \lambda - 3k_2 + \lambda k_2 - 3\lambda + \lambda^2 + 2 - k_1 \tag{51}$$

$$= \lambda^2 + (k_2 - 4)\lambda + (-k_1 - 3k_2 + 5) \tag{52}$$

Thus, $C = 1$, $D = k_2 - 4$, and $E = -k_1 - 3k_2 + 5$.

(c) (8 pts.) With a discrete system different from that in part (b), we assume a fixed value of k_2 such that our characteristic polynomial becomes the following:

$$\lambda^2 - 2\lambda - (k_1 + 2)$$

Find a range of values for k_1 for which the system is stable. Please write your answers as either a number or interval(s) of number(s) (i.e. 8 inclusive to ∞ would be $[8, \infty)$); if there is no solution, you may say so. **Justify your answer with your work.**

Solution: Using the quadratic formula, we can solve for the values of λ.

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-(k_1+2))}}{2(1)} \tag{53}$$

$$= 1 \pm \frac{\sqrt{4 + 4k_1 + 8}}{2} \tag{54}$$

$$= 1 \pm \frac{\sqrt{12 + 4k_1}}{2} \tag{55}$$

$$= 1 \pm \sqrt{3 + k_1} \tag{56}$$

There are three cases of interest, when the expression under the square root, $3 + k_1$, is positive, zero, or negative.

- If $3 + k_1 > 0$ then $\sqrt{3 + k_1} > 0$. This means there is one eigenvalue, strictly greater than 1: $\lambda_+ = 1 + \sqrt{3 + k_1} > 1$. Thus, $|\lambda_+| \geq 1$ leading to instability for values of k_1 with $3 + k_1 > 0$, i.e., for k_1 such that $-3 < k_1$.
- If $3 + k_1 = 0$, then $\lambda = 1 \pm \sqrt{0} = 1$. Thus, for both eigenvalues, $|\lambda| \geq 1$, leading to instability for $k_1 = -3$.
- If $3 + k_1 < 0$, then we have complex eigenvalues. Since $3 + k_1 = -|3 + k_1|$, we can write that $\lambda = 1 \pm \sqrt{-|3 + k_1|} = 1 \pm \mathrm{j}\sqrt{|3 + k_1|}$. The magnitude of both eigenvalues is given by $|\lambda| = \sqrt{1^2 + (3 + k_1)^2} \geq 1$ since $3 + k_1$ is nonzero. So for k_1 such that $k_1 < -3$, we have instability.

Any choice of $k_1 \in \mathbb{R}$ will lead to an unstable system since at least one of eigenvalues will have $|\lambda| \geq 1$. Thus, there is no solution.

Caveat: Students can also acknowledge that for a closed-loop system, since we effectively do not have to worry about an external input causing our system to grow unbounded, $|\lambda| = 1$ is valid to keep the system stable. In this case, students must determine that the only solution for stability is $k_1 = 3$ based on the previous analysis.

9. Continuous Time Discretization and Back (14 pts.)

In this problem we will examine how to perform system ID on a continuous time system. Consider a car with a two-dimensional state, $\vec{x}$, whose dynamics are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{57}$$

where $A \in \mathbb{R}^{2\times 2}$ and $\vec{b} \in \mathbb{R}^2$ are unknown. The state's entries are written as $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(a) (6 pts.) Suppose we discretized our continuous time system and obtained the following dynamics for our system with state $\vec{x}_d[i] = \begin{bmatrix} x_{d,1}[i] \\ x_{d,2}[i] \end{bmatrix}$.

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{58}$$

The parameters $A_d \in \mathbb{R}^{2\times 2}$ and $\vec{b}_d \in \mathbb{R}^2$ are unknown. We apply discrete input $u_d[i]$ from $i = 0$ to $i = 3$, and obtain the following state observations.

i	$u_d[i]$	$x_{d,1}[i]$	$x_{d,2}[i]$
0	-1	2	3
1	0	7	8
2	1	4	6
3	0	5	9
4	N/A	8	13

Figure 5: Data Collected from Sampling the System

We want to identify A_d and $\vec{b}_d$ by setting up a least squares problem of the form $DP \approx S$ where $P = \begin{bmatrix} A_d^\top \\ \vec{b}_d^\top \end{bmatrix}$.

Express the D and S matrices in terms of numerical values of $x_{d,1}[i]$, $x_{d,2}[i]$, and $u_d[i]$ from the table.

Solution: Note that we can rewrite eq. (58) as

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] = \begin{bmatrix} A_d & \vec{b}_d \end{bmatrix} \begin{bmatrix} \vec{x}_d[i] \\ u_d[i] \end{bmatrix} \tag{59}$$

Taking transposes of both sides, we have

$$\vec{x}_d[i+1]^\top = \begin{bmatrix} \vec{x}_d[i]^\top & u_d[i] \end{bmatrix} \begin{bmatrix} A_d^\top \\ \vec{b}_d^\top \end{bmatrix} \tag{60}$$

Now, we have to use the known quantities, i.e., $\vec{x}_d[i]$ and $u_d[i]$, to estimate A_d and $\vec{b}_d$. We know this equation holds for $i = 0, \ldots, 3$ so we will "stack" the equations together as follows:

$$\begin{bmatrix} \vec{x}_d[4] \\ \vec{x}_d[3] \\ \vec{x}_d[2] \\ \vec{x}_d[1] \end{bmatrix} = \begin{bmatrix} \vec{x}_d[3] & u_d[3] \\ \vec{x}_d[2] & u_d[2] \\ \vec{x}_d[1] & u_d[1] \\ \vec{x}_d[0] & u_d[0] \end{bmatrix} \begin{bmatrix} A_d^\top \\ \vec{b}_d^\top \end{bmatrix} \tag{61}$$

$$\begin{bmatrix} x_{d,1}[4] & x_{d,2}[4] \\ x_{d,1}[3] & x_{d,2}[3] \\ x_{d,1}[2] & x_{d,2}[2] \\ x_{d,1}[1] & x_{d,2}[1] \end{bmatrix} = \begin{bmatrix} x_{d,1}[3] & x_{d,2}[3] & u_d[3] \\ x_{d,1}[2] & x_{d,2}[2] & u_d[2] \\ x_{d,1}[1] & x_{d,2}[1] & u_d[1] \\ x_{d,1}[0] & x_{d,2}[0] & u_d[0] \end{bmatrix} \begin{bmatrix} A_d^\top \\ \vec{b}_d^\top \end{bmatrix} \tag{62}$$

Substituting values in, we get that D and S are:

$$\underbrace{\begin{bmatrix} 5 & 9 & 0 \\ 4 & 6 & 1 \\ 7 & 8 & 0 \\ 2 & 3 & -1 \end{bmatrix}}_{D} P = \underbrace{\begin{bmatrix} 8 & 13 \\ 5 & 9 \\ 4 & 6 \\ 7 & 8 \end{bmatrix}}_{S} \tag{63}$$

Note that any permutations of these rows would constitute an acceptable answer, provided that the rows of both D and S are permuted in the same way.

(b) (3 pts.) Suppose now that you want to solve for an estimate of A, the matrix in the continuous system, using your estimate of A_d, the matrix in the discretized system. We will try this by first looking at a scalar system.

Consider the scalar continuous time system in eq. (64) and its corresponding discretization with $x_d[i] = x(i\Delta)$ and $u_d[i] = u(i\Delta)$ in eq. (65).

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \underbrace{\lambda}_{a} x(t) + bu(t) \tag{64}$$

$$x_d[i+1] = \underbrace{e^{\lambda\Delta}}_{a_d} x_d[i] + \underbrace{\frac{e^{\lambda\Delta} - 1}{\lambda} b}_{b_d} u_d[i] \tag{65}$$

You know $a_d \approx e^{\lambda\Delta}$ by system identification. **Express a in terms of a_d and Δ.**

Solution: We know that, in this instance, $\lambda = a$. Hence,

$$a_d = e^{a\Delta} \tag{66}$$

$$\ln(a_d) = a\Delta \tag{67}$$

$$a = \frac{\ln(a_d)}{\Delta} \tag{68}$$

(c) (5 pts.) Let's return to the original matrix-vector system. It is true that if A can be diagonalized as $A = V\Lambda V^{-1}$, then $A_d = V\Lambda_d V^{-1}$. Suppose Λ_d has entries $\begin{bmatrix} a_{d,1} & 0 \\ 0 & a_{d,2} \end{bmatrix}$. **Using the entries of Λ_d, solve for Λ, the matrix of eigenvalues of A. Then, express A in terms of V, V^{-1}, $a_{d,1}$, $a_{d,2}$, and Δ.**

(HINT: Use your result from part (b) to express the entries of Λ in terms of the entries of Λ_d.)

Solution: We will define $\vec{y}(t)$ to be the representation of $\vec{x}(t)$ in the eigenbasis of A_d. In other words, $\vec{y}(t) = V^{-1}\vec{x}(t)$. Similarly, we have that $\vec{y}_d[i] = V^{-1}\vec{x}_d[i]$. By applying diagonalization to this problem, we have the following two continuous and discrete time systems respectively:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y}(t) = \Lambda\vec{y}(t) + V^{-1}\vec{b}u(t) \tag{69}$$

$$\vec{y}_d[i+1] = \Lambda_d \vec{y}_d[i] + V^{-1}\vec{b}_d u_d[i] \tag{70}$$

We know Λ_d, and we know it is diagonal, so we can view this as two separate, scalar system ID problems. We can denote $\Lambda = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}$, $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, and $\vec{y}_d[i] = \begin{bmatrix} y_{d,1}[i] \\ y_{d,2}[i] \end{bmatrix}$. The two system ID problems would be

$$\frac{\mathrm{d}}{\mathrm{d}t} y_1(t) = \Lambda_{11} y_1(t) + \left(V^{-1}\vec{b}\right)_1 u(t) \tag{71}$$
$$y_{d,1}[i+1] = a_{d,1} y_{d,1}[i] + \left(V^{-1}\vec{b}_d\right)_1 u_d[i] \tag{72}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} y_2(t) = \Lambda_{22} y_2(t) + \left(V^{-1}\vec{b}\right)_2 u(t) \tag{73}$$
$$y_{d,2}[i+1] = a_{d,2} y_{d,2}[i] + \left(V^{-1}\vec{b}_d\right)_2 u_d[i] \tag{74}$$

Pattern matching the results from the previous part, we have

$$\Lambda = \begin{bmatrix} \frac{\ln\left(a_{d,1}\right)}{\Delta} & 0 \\ 0 & \frac{\ln\left(a_{d,2}\right)}{\Delta} \end{bmatrix} \tag{75}$$

Thus,

$$A = V \begin{bmatrix} \frac{\ln\left(a_{d,1}\right)}{\Delta} & 0 \\ 0 & \frac{\ln\left(a_{d,2}\right)}{\Delta} \end{bmatrix} V^{-1} \tag{76}$$

