EECS 16B DIS 11B Gaoyue

Singular Value Decomposition.

Every matrix $A \in \mathbb{R}^{m \times n}$ $\quad \text{rank}(A) = r$

$$A = \underbrace{U}_{m \times m} \underbrace{\Sigma}_{m \times n} \underbrace{V^T}_{n \times n} \qquad \Sigma = \begin{bmatrix} \boxed{\begin{matrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{matrix}}_{\Sigma_r} & 0 \\ 0 & 0 \end{bmatrix}$$

- U and V are orthogonal matrices
- Σ_r is diagonal

$$\sigma_i^2 = \lambda_i(A^TA) = \lambda_i(AA^T) \quad i = 1, \dots, r$$

- $\vec{v}_i$: eigenvector of A^TA ←

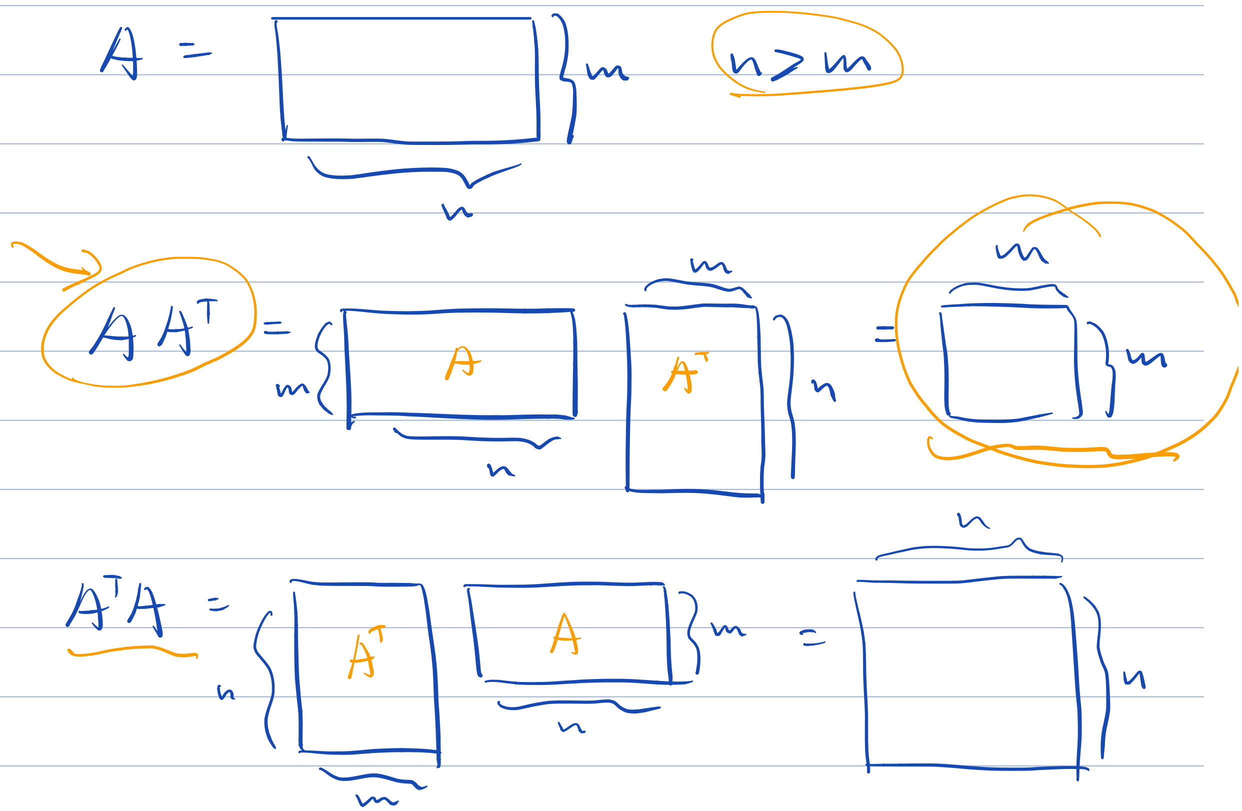
$\vec{u}_i$: eigenvector of AA^T

- $N(A)$ spanned by V_{n-r}
- $R(A)$ spanned by U_r

1. Understanding the SVD assume rank$(A) = m$

We can compute the SVD for a wide matrix A with dimension $m \times n$ where $n > m$ using $A^\top A$ with the method covered in lecture. However, when doing so, you may realize that $A^\top A$ is much larger than $AA^\top$ for such wide matrices. This makes it more efficient to find the eigenvalues for $AA^\top$. In this question, we will explore how to compute the SVD using $AA^\top$ instead of $A^\top A$.

(a) What are the dimensions of $AA^\top$ and $A^\top A$?



(b) Given that the SVD of A is $A = U\Sigma V^\top$, find a symbolic expression for $AA^\top$.

$$A = U\Sigma V^T$$

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T$$

$$= U\Sigma V^T V \Sigma^T U^T$$

$$AA^T = U\Sigma\Sigma^T U^T$$

(c) Using the solution to the previous part, how can we find U and Σ from $AA^\top$?

$$\Sigma\Sigma^T = \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_m & \end{bmatrix}}_{n} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \\ \hline & 0 & \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_m^2 \end{bmatrix}$$

$$AA^T\vec{u}_i = U\Sigma\Sigma^T U^T \vec{u}_i \qquad i = 1, \dots, m$$

$$= U \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_m^2 \end{bmatrix} \vec{e}_i = U(\sigma_i^2 \vec{e}_i)$$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ entry} \qquad U = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}$$

$$= \sigma_i^2 \vec{u}_i$$

$e_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\vec{u}_i$ $i = 1, \dots, m$ are eigenvectors of AA^T with eigenvalues σ_i^2

(d) Now that we have found the singular values σ_i and the corresponding vectors $\vec{u}_i$ in the matrix U, can you find the corresponding vectors $\vec{v}_i$ in V?

$$AA^T\vec{u}_i = \sigma_i^2 \vec{u}_i$$

$$A^T AA^T \vec{u}_i = A^T \sigma_i^2 \vec{u}_i$$

$$A^TA(A^T\vec{u}_i) = \sigma_i^2 A^T\vec{u}_i$$

$\Rightarrow$ $A^T\vec{u}_i$ is an eigenvector of A^TA with eigenvalue σ_i^2

define $\vec{v}_i = \dfrac{A^T\vec{u}_i}{\|A^T\vec{u}_i\|}$

$$\vec{v}_i = \frac{A^T\vec{u}_i}{\|A^T\vec{u}_i\|}$$

$$= \frac{A^T \vec{u}_i}{\sqrt{\|A^T \vec{u}_i\|^2}}$$

$$= \frac{A^T \vec{u}_i}{\sqrt{(A^T \vec{u}_i)^T (A^T \vec{u}_i)}}$$

$$= \frac{A^T \vec{u}_i}{\sqrt{\vec{u}_i^T A A^T \vec{u}_i}}$$

$$= \frac{A^T \vec{u}_i}{\sqrt{\vec{u}_i^T \sigma_i^2 \vec{u}_i}}$$

$$= \frac{A^T \vec{u}_i}{\sqrt{\sigma_i^2 \|\vec{u}_i\|^2}} \quad = 1$$

$$\boxed{\vec{v}_i = \frac{A^T \vec{u}_i}{\sigma_i}}$$

$\vec{v}_i^T \vec{v}_j$

(e) Now we have a way to find the vectors $\vec{v}_i$ in matrix V! Verify that these vectors are orthonormal.

$$\vec{v}_i^T \vec{v}_j = \frac{\vec{u}_i^T A}{\sigma_i} \frac{A^T \vec{u}_j}{\sigma_j}$$

$$= \frac{\vec{u}_i (A A^T \vec{u}_j)}{\sigma_i \sigma_j} \quad \sigma_j^2 \vec{u}_j$$

$$= \frac{\sigma_j^2 \, \vec{u}_i^{\top} \vec{u}_j}{\sigma_i \sigma_j}$$

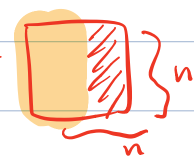
$$\vec{v}_i^{\top} \vec{v}_j = \frac{\sigma_j}{\sigma_i} \vec{u}_i^{\top} \vec{u}_j$$

when $i = j$, $\vec{v}_i^{\top} \vec{v}_j = 1$ ✓

$i \neq j$, $\vec{v}_i^{\top} \vec{v}_j = 0$ ✓

$AA^{\top} \in \mathbb{R}^{m \times m}$

$\underset{i=1,\dots,m}{\vec{u}_i} \longrightarrow \vec{v}_i = \dfrac{A^{\top} \vec{u}_i}{\sigma_i}$, $i = 1, \dots, m$

$V =$  n, $n > m$

(f) Now that we have found $\vec{v}_i$, you may notice that we only have $m < n$ vectors of dimension n. This is not enough for a basis. How would you complete the m vectors to form an orthonormal basis?

Gram-Schmidt

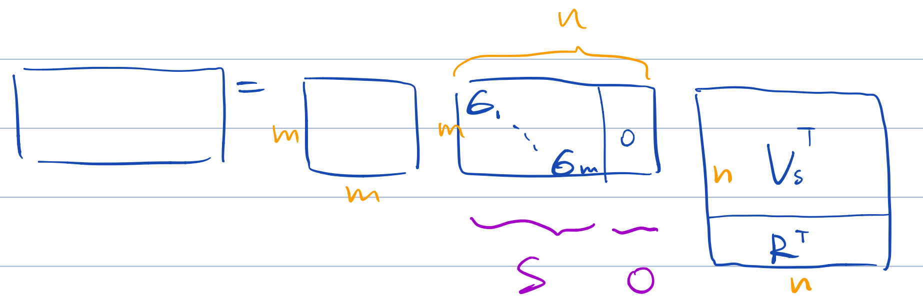
(g) (Practice.) Given that $A = U\Sigma V^{\top}$ verify that the vectors you found to extend the $\vec{v}_i$ into a basis are in the nullspace of A.

Null(A) is spaned by

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{\begin{matrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{matrix}}_{V_s} & \underbrace{\begin{matrix} | & & | \\ \vec{v}_{m+1} & \cdots & \vec{v}_n \\ | & & | \end{matrix}}_{R} \end{bmatrix} = \begin{bmatrix} \underbrace{V_s}_{n \times m} & \underbrace{R}_{n \times (n-m)} \end{bmatrix}$$

$$A = U \Sigma V^T$$



$$\Sigma = \begin{bmatrix} \underset{m \times m}{S} & \underset{m \times (n-m)}{0} \end{bmatrix}$$

$$AR = U \Sigma V^T R$$

$$= U \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} V_s^T \\ R^T \end{bmatrix} R$$

$$= U \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} V_s^T R \\ R^T R \end{bmatrix}$$

$$= U \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} 0 \\ R^T R \end{bmatrix}$$

$$= U \begin{bmatrix} 0 \end{bmatrix} = 0$$

$\operatorname{rank}(A) = r$

(h) Using the previous parts of this question and what you learned from lecture, write out a procedure on how to find the SVD for *any* matrix.

Pick A^TA or AA^T

AA^T $\qquad$ AA^T

If A^TA: find eigenvalues λ_i of A^TA
and order them $\lambda_1 \geq \cdots \geq \lambda_r > 0$

find eigenvectors $\vec{v}_i$ of A^TA

$\vec{u}_i$ $\qquad$ AA^T

i.e $A^TA\vec{v}_i = \lambda_i \vec{v}_i$ $\quad i = 1, \dots, r$

$AA^T\vec{u}_i = \lambda_i \vec{u}_i$

Set $\sigma_i = \sqrt{\lambda_i}$

get $\vec{u}_i$ by $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$ $\quad i = 1, \dots, r$

$\vec{v}_i$ $\qquad$ $\vec{v}_i = \frac{1}{\sigma_i} A^T\vec{u}_i$

Complete U by Gram-Schmidt

V