EECS 16B DIS 12B Gaoyue

Geometric interpretation of SVD

$A \in \mathbb{R}^{2\times 2}$

$A = U \Sigma V^T$

① U, V are orthogonal matrices

$\Rightarrow$ U, V preserves the norm of vectors

$\Rightarrow$ U, V rotate & reflect vectors

② Σ is diagonal

$\Rightarrow$ Σ "stretches" vectors
horizontally (by σ_1) and
vertically (by σ_2)

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

1. Geometric interpretation of the SVD

In this exercise, we explore the geometric interpretation of symmetric matrices and how this connects to the SVD. We consider how a real 2×2 matrix acts on the unit circle, transforming it into an ellipse. It turns out that the principal semiaxes of the resulting ellipse are related to the singular values of the matrix, as well as the vectors in the SVD.

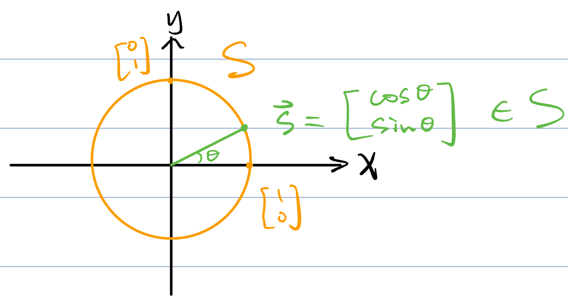
(a) Consider the real 2×2 matrix

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}. \tag{1}$$

Also consider the unit circle in $\mathbb{R}^2$,

$$S = \left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \middle| 0 \leq \theta < 2\pi \right\}. \tag{2}$$

Plot the transformed circle, AS, on the $\mathbb{R}^2$ plane.



$$A\vec{s} = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

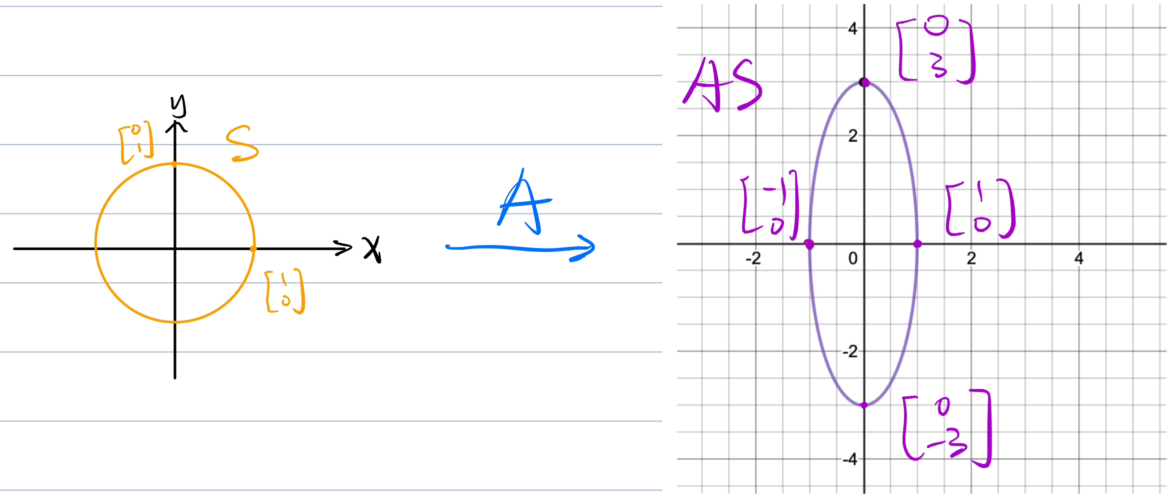
$$= \begin{bmatrix} -\sin\theta \\ 3\cos\theta \end{bmatrix}$$

$$\Rightarrow AS = \left\{ \begin{bmatrix} -\sin\theta \\ 3\cos\theta \end{bmatrix} \middle| 0 \leq \theta \leq 2\pi \right\}$$

Some points in AS:

$\theta = 0 \Rightarrow \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $\qquad$ $\theta = \frac{\pi}{2} \Rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\theta = \pi \Rightarrow \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ $\theta = \frac{3}{2}\pi \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



(b) Now let's consider how this transformation looks in the lens of the SVD. The SVD for matrix A is:

$$A = U\Sigma V^\top = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3}$$

$$A\vec{x} = U\Sigma V^\top \vec{x} = U\left(\Sigma\left(V^\top \vec{x}\right)\right). \tag{4}$$

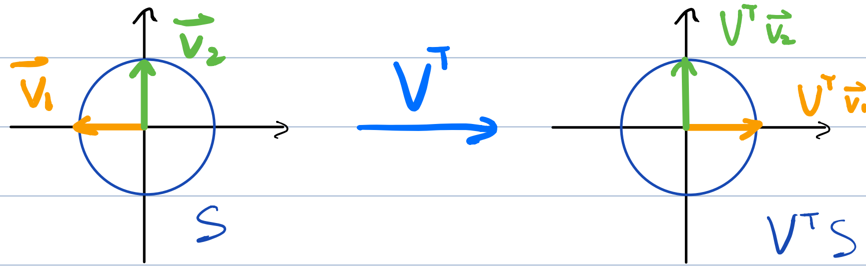
Let's start by examining the effects of each of these matrices one at a time, right to left, in the same order that they would be applied to a vector $\vec{x}$.

What does the unit circle look like after being transformed by just $V^\top$? Plot $S_1 = V^\top S$ on the $\mathbb{R}^2$ plane. Geometrically speaking, what does $V^\top$ do to any given $\vec{x}$.

$\vec{s} = \begin{bmatrix} s_x \\ s_y \end{bmatrix} \in S$

$V^\top \vec{s} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \vec{s} = \begin{bmatrix} -s_x \\ s_y \end{bmatrix}$

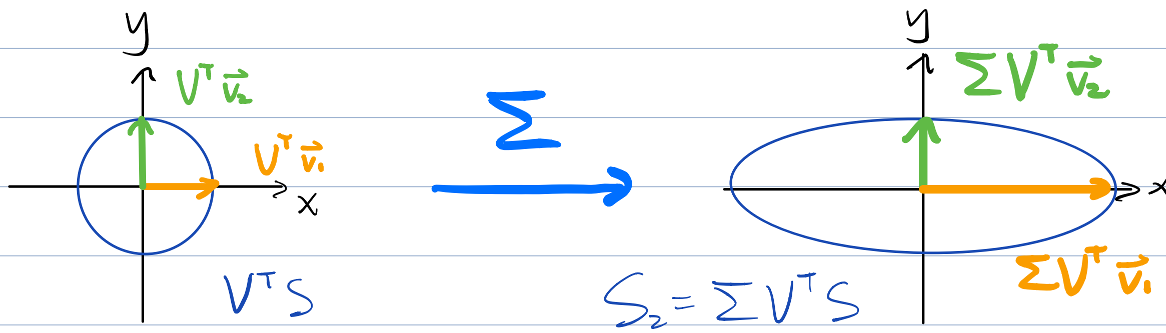
$V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$$\vec{v}_i \xrightarrow{V^\top} \vec{e}_i$$

(c) What does the unit circle look like after being transformed by $\Sigma V^\top$? Plot $S_2 = \Sigma V^\top S$ on the $\mathbb{R}^2$ plane. Geometrically speaking, what is the Σ matrix doing to any given $V^\top \vec{x}$?

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$



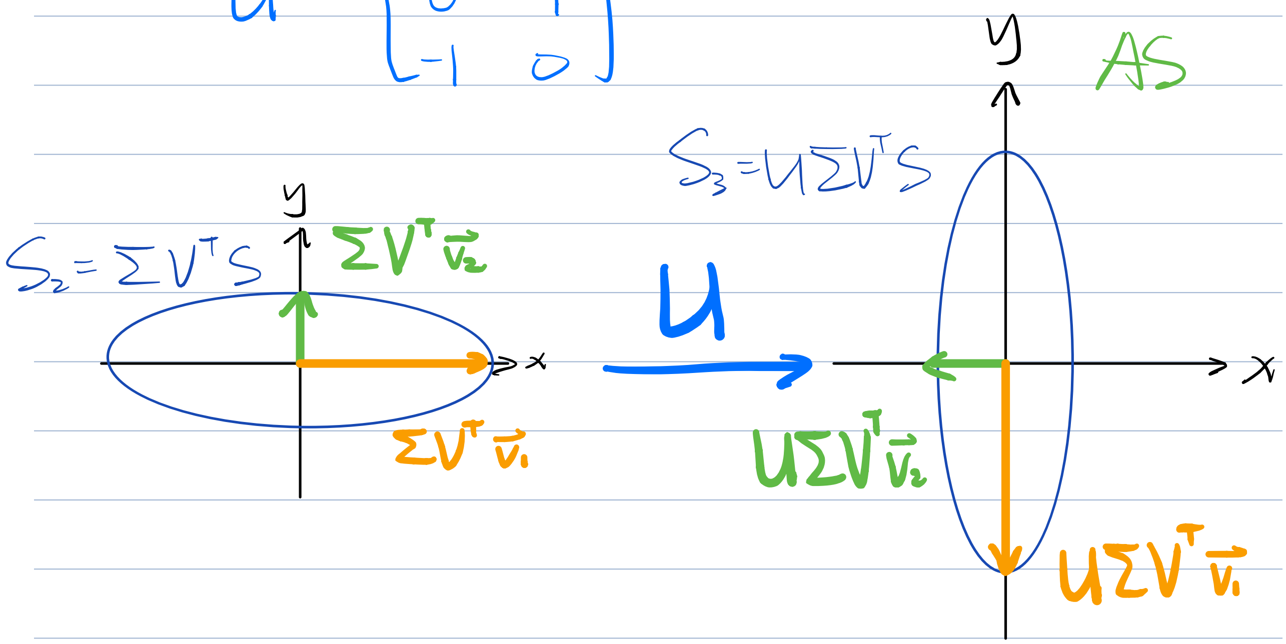
$V^\top \vec{x}$: coordinates of $\vec{x}$ in V-basis

Σ scales $\vec{x}$ in the direction of $\vec{v}_1$ by $\sigma_1 = 3$

of $\vec{v}_2$ by $\sigma_2 = 1$

(d) What does the unit circle look like after being transformed by $U\Sigma V^\top$? Plot $S_3 = U\Sigma V^\top S$ on the $\mathbb{R}^2$ plane. Geometrically speaking, what is the U matrix doing to any given $\Sigma V^\top \vec{x}$?

$U = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$



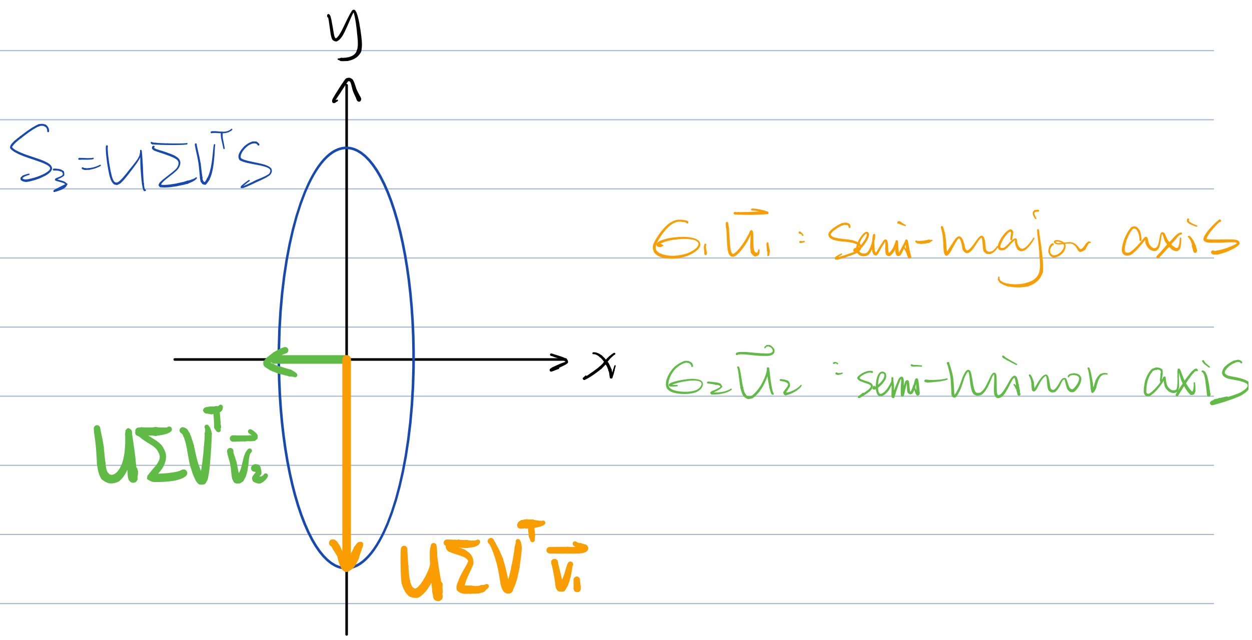
(e) Consider the columns of the matrices U, V from the SVD of A in part (b), and treat them as vectors in $\mathbb{R}^2$. Let $U = (\vec{u_1}\ \vec{u_2})$, $V = (\vec{v_1}\ \vec{v_2})$. Let σ_1, σ_2 be the singular values of A, where $\sigma_1 \geq \sigma_2$. In your plot of AS, draw the vectors $\sigma_1\vec{u_1}$ and $\sigma_2\vec{u_2}$ from the origin. What do these vectors correspond to geometrically?

$$A = U\Sigma V^T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U\Sigma V^T \vec{v}_1 = U\Sigma \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3\vec{u}_1 = \sigma_1 \vec{u}_1$$

$$U\Sigma V^T \vec{v}_2 = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\,\vec{u}_2 = \sigma_2 \vec{u}_2$$



(f) Repeat parts (b-e) for the following matrices, and note down any interesting things you notice.

i. A rotation matrix, $A_1 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

ii. A diagonal matrix, $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

iii. A symmetric matrix, $A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

iv. A matrix with non-trivial nullspace, $A_4 = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$.

v. An arbitrary matrix, $A_5 = \begin{bmatrix} 1.6 & 2.4 \\ -0.4 & -1 \end{bmatrix}$.