EECS 16B DIS 2B Gaoyue

1. RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor over time in an RC circuit. We set up our problem by first defining four functions over time: $I(t)$ is the current at time t, $V(t)$ is the voltage across the circuit at time t, $V_R(t)$ is the voltage across the resistor at time t, and $V_C(t)$ is the voltage across the capacitor at time t.

Recall from 16A that the voltage across a resistor is defined as $V_R = RI_R$ where I_R is the current across the resistor. Also, recall that the voltage across a capacitor is defined as $V_C = \frac{Q}{C}$ where Q is the charge across the capacitor.

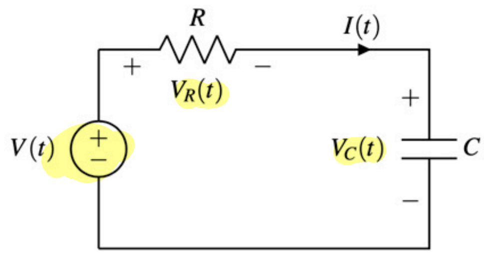


Figure 1: Example Circuit

(a) First, find an equation that relates the current across the capacitor $I(t)$ with the voltage across the capacitor $V_C(t)$.

$$Q(t) = C V_c(t)$$

$$V_c(t) = \frac{Q(t)}{C}$$

$$\frac{dV_c(t)}{dt} = \frac{1}{C} \frac{dQ(t)}{dt}$$

$$\rightarrow \quad \frac{d}{dt} V_c(t) = \frac{1}{C} I(t)$$

(b) Write a system of equations that relates the functions $I(t)$, $V_C(t)$, and $V(t)$.

From KVL, $V(t) - V_R(t) - V_c(t) = 0$

$$V(t) - I(t)R - V_c(t) = 0$$

$$\rightarrow \quad R I(t) + V_c(t) = V(t)$$

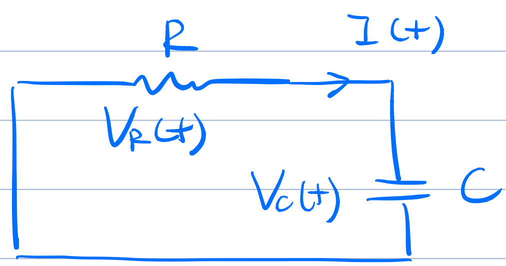
(c) So far, we have three unknown functions and only one equation, but we can remove $I(t)$ from the equation using what we found in part (a). Rewrite the previous equation in part (b) in the form of a differential equation.

From a),

$$I(t) = \frac{dV_c(t)}{dt} C$$

$$RC \frac{dV_c(t)}{dt} + V_c(t) = V(t)$$

(d) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0$ for all $t \geq 0$. Solve the differential equation for $V_C(t)$ for $t \geq 0$.



$$RC \frac{dV_c(t)}{dt} + V_c(t) = \cancel{V(t)}^{\,0}$$

$$\frac{dV_c(t)}{dt} = -\frac{1}{RC} V_c(t)$$

$$\frac{dV(t)}{dt} = a V(t)$$

$$\frac{d}{dt}(Ae^{bt})$$

Guess: $V_c(t) = Ae^{bt}$ $\leftarrow$ $= Abe^{bt}$

$b = -\frac{1}{RC}$ $\qquad = b\,(Ae^{bt})$

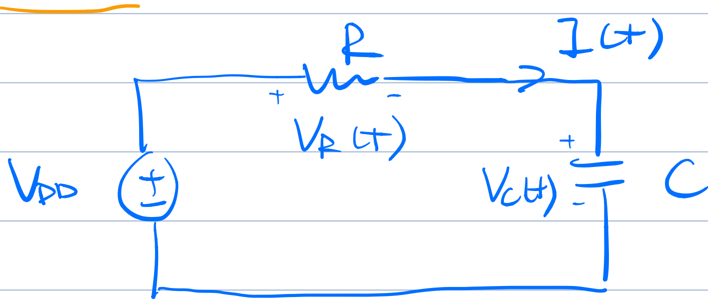
Now solve A using initial condition.

$\rightarrow$ $V_c(0) = V_{DD}$

$V_c(0) = Ae^{b \cdot 0} = A = V_{DD}.$

$$\boxed{V_C(t) = V_{DD}\, e^{-\frac{1}{RC}t}}$$

(e) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Solve the differential equation for $V_C(t)$ for $t \geq 0$.



$$\rightarrow \quad RC\frac{dV_C(t)}{dt} + V_C(t) = \underbrace{V(t)}\qquad V_{DD}$$

$$\frac{dV_C(t)}{dt} = \frac{V_{DD} - V_C(t)}{RC}$$

$$= -\frac{1}{RC}\boxed{(\underbrace{V_C(t) - V_{DD}}_{\tilde{V}_C(t)})}$$

$$\frac{dx(t)}{dt} = ax(t) + b$$

$$= a\left(x(t) + \frac{b}{a}\right)$$

$$V_C(t) = V_{DD} + \tilde{V}_C(t)$$

$$\frac{d}{dt}(\underbrace{V_C(t) - V_{DD}}_{\tilde{V}_C(t)}) = \frac{d}{dt}V_C(t) - \cancel{\frac{d}{dt}V_{DD}}^{\,0}$$

$$\frac{d}{dt}\tilde{V}_c(t) = \frac{d}{dt}V_c(t)$$

$$\Rightarrow \quad \frac{d\tilde{V}_c(t)}{dt} = -\frac{1}{RC}\tilde{V}_c(t)$$

$$\Rightarrow \quad \tilde{V}_c(t) = Be^{-\frac{1}{RC}t}$$

Solve B.

$\tilde{V}_c(0)$

I.C. $\quad V_c(0) = 0$

$$\tilde{V}_c(0) = 0 - V_{DD}$$

$$\tilde{V}_c(0) = -V_{DD}$$

$$\tilde{V}_c(0) = Be^0 = B = -V_{DD}$$

$$\tilde{V}_c(t) = -V_{DD}e^{-\frac{1}{RC}t}$$

$$V_c(t) = V_{DD} + \tilde{V}_c(t)$$

$$= V_{DD}(1 - e^{-\frac{1}{RC}t})$$