EECS 16B DIS 3B Gaoyue

1. Changing Coordinates and Systems of Differential Equations

Suppose we have the pair of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}x_1(t) = -9x_1(t)$$

$$\frac{d}{dt}x_2(t) = -2x_2(t)$$

with initial conditions $x_1(0) = -1$ and $x_2(0) = 3$.

(a) Solve for $x_1(t)$ and $x_2(t)$ for $t \geq 0$.

$$x_1(t) = k_1 e^{-9t}$$

$$x_2(t) = k_2 e^{-2t}$$

Solve k_1, k_2, using I.C.

$$x_1(0) = k_1 e^0 = k_1 = -1$$

$$x_2(0) = k_2 e^0 = k_2 = 3$$

$$x_1(t) = -e^{-9t}$$

$$x_2(t) = 3e^{-2t}$$

Suppose we are actually interested in a different set of variables with the following differential equations:

$$\frac{d}{dt}z_1(t) = -5z_1(t) + 2z_2(t)$$

$$\frac{d}{dt}z_2(t) = 6z_1(t) - 6z_2(t).$$

(b) Write out the above system of differential equations in matrix form. Assuming that the initial state $\vec{z}(0) = \begin{bmatrix} 7 & 7 \end{bmatrix}^T$, can we solve this system directly?

$$\rightarrow \quad \frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

original $\rightarrow$ $\frac{d}{dt}x(t) = ax(t) + b$

$\rightarrow \frac{d}{dt}\vec{z}(t) = A\vec{z}(t)$

(c) Consider that in our frustration with the previous system of differential equations, we start hearing voices. These voices whisper to us that that we should try the following change of variables:

$$z_1(t) = -y_1(t) + 2y_2(t)$$
$$z_2(t) = 2y_1(t) + 3y_2(t).$$

Write out this transformation in matrix form ($\vec{z} = V\vec{y}$).

$$\underbrace{\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}}_{\vec{z}} = \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}_{\vec{y}}$$

$$V^{-1}\vec{z} = \underbrace{V^{-1}V}_{I}\vec{y}$$

$$\vec{y} = V^{-1}\vec{z}$$

For each of the parts (d) - (f), solve the questions two ways: 1. using direct substitution, and 2. using matricies and vectors .

(d) How do the initial conditions for $z_i(t)$ translate into the initial conditions for $y_i(t)$? $\vec{z}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

1. By direct substitution:

$$\begin{cases} z_1(0) = -y_1(0) + 2y_2(0) = 7 \\ z_2(0) = 2y_1(0) + 3y_2(0) = 7 \end{cases}$$

$y_1(0) = -1$

$y_2(0) = 3$

2. Using matrices and vectors:

$\vec{z} = V\vec{y}$

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \overset{V}{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = V^{-1} \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

For a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

its inverse :

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

$$\vec{y}(0) = V^{-1} \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(e) Rewrite the differential equations in terms of $y_i(t)$. Can we solve this system of differential equations?

1. By direct substitution:

2. Using matrices and vectors:

$$\frac{d\vec{z}}{dt} = A\vec{z}$$

$$\vec{z} = V\vec{y} \qquad \vec{y} = V^{-1}\vec{z}$$

Goal: find B s.t. $\frac{d\vec{y}}{dt} = B\vec{y}$

$$\frac{d\vec{y}}{dt} = \frac{d}{dt}(\overbrace{V^{-1}\vec{z}}^{\vec{y}})$$

$$= V^{-1}\frac{d}{dt}\vec{z}$$

$$= V^{-1}A\vec{z}$$

LHS

$$\boxed{\frac{d\vec{y}}{dt} = V^{-1}AV\vec{y}}$$

$$B = V^{-1}AV$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}}_{B \text{ is diagonal}} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\begin{cases} \underline{\frac{d}{dt} y_1(t) = -1\, y_1(t)} \\ \underline{\frac{d}{dt} y_2(t) = -2\, y_2(t)} \end{cases}$$

$$y_1(t) = K_1 e^{-1t} \qquad y_1(0) = -1$$
$$y_2(t) = K_2 e^{-2t} \qquad y_2(0) = 3$$

$$\Rightarrow \boxed{\begin{aligned} y_1(t) &= -e^{-1t} \\ y_2(t) &= 3e^{-2t} \end{aligned}}$$

f) Solve it in terms of $\vec{z}$

$$\vec{z} = V\vec{y}$$

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-1t} \\ 3e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-1t} + 6e^{-2t} \end{bmatrix}$$

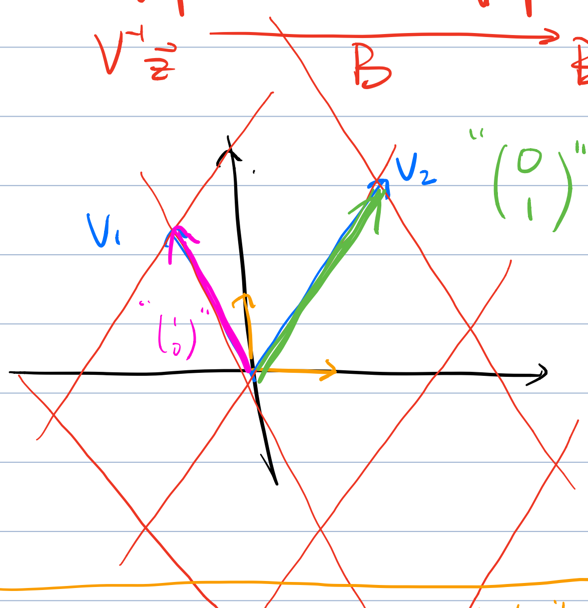
$$\begin{bmatrix} -2e^{-3t} + 9e^{-2t} \end{bmatrix}$$

$$B = V^{-1}AV = \begin{bmatrix} * & & 0 \\ & \times & \\ & & \times \\ 0 & & & \times \end{bmatrix}$$

$$A = VBV^{-1}$$

$$= \underbrace{\begin{bmatrix} & \\ & \end{bmatrix}}\begin{bmatrix} \times & & \\ & \times & \\ & & \times \end{bmatrix}\underbrace{\begin{bmatrix} & \\ & \end{bmatrix}}$$

$$\begin{array}{llll} \text{Standard basis} & \vec{z} & \xrightarrow{\quad A \text{ "} \underset{\sim}{VBV^{-1}} \text{"} \quad} & A\vec{z} \text{ "} \frac{d}{dt}\vec{z} \text{"} \\ & V^{-1}\downarrow\uparrow V & & V^{-1}\downarrow\uparrow V \\ \text{eigenbasis} & V^{-1}\vec{z} & \xrightarrow{\quad B \quad} & BV^{-1}\vec{z} \end{array}$$



$\quad v_1 \quad v_2$

$$\underset{\sim}{V} = \begin{bmatrix} \begin{pmatrix}-1\\2\end{pmatrix} & \begin{pmatrix}2\\3\end{pmatrix} \end{bmatrix}$$

$$V\vec{x}$$

$$\underset{\sim}{V}\begin{bmatrix}1\\0\end{bmatrix} = \underset{\sim}{\begin{bmatrix}-1\\2\end{bmatrix}}$$

$$V\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$$

Apologies but v_1 is the "$\begin{pmatrix}1\\0\end{pmatrix}$" vector in the eigenbasis and v_2 is the $\begin{pmatrix}0\\1\end{pmatrix}$ in the eigenbasis