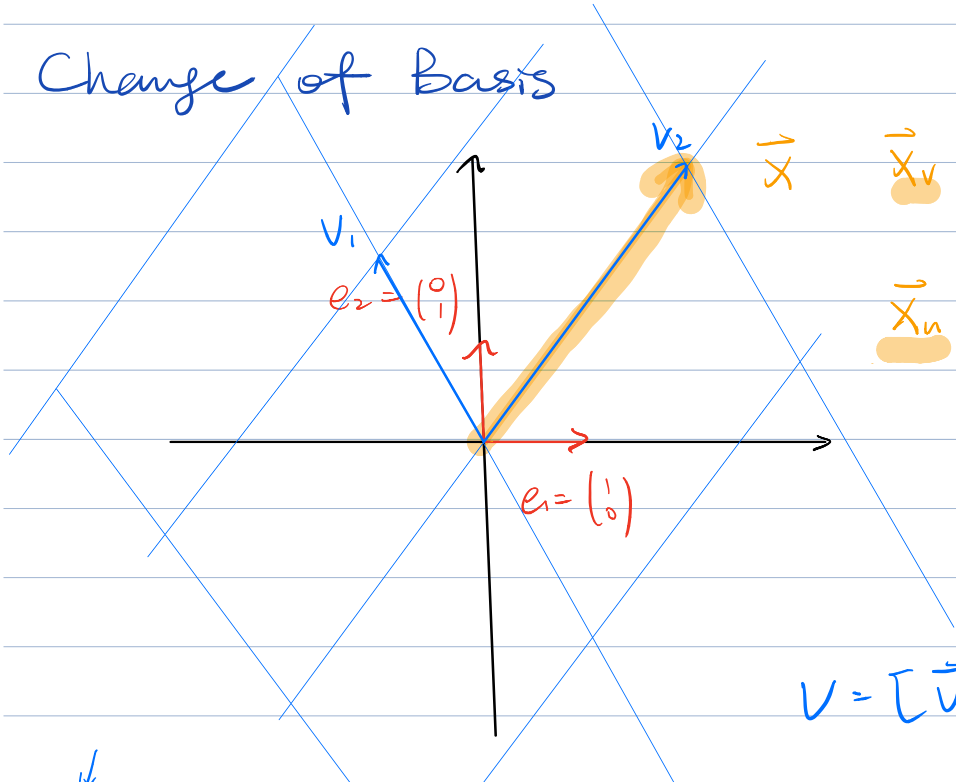
EECS 16B DIS 4B Caoyue

Change of Basis



$\vec{x}$ $\quad \vec{x}_v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{x}_n = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$V = [\vec{v}_1 \ \ \vec{v}_2]$

$$V = [\vec{v}_1 \ \ \vec{v}_2] = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$V\vec{x}_v = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \vec{x}$$

$$A = V \Lambda V^{-1}$$

$$\begin{tikzcd}
\text{Standard basis} & \vec{x} \arrow[rr, "A"] \arrow[d, shift right, "V^{-1}"'] & & A\vec{x} = V\Lambda V^{-1}\vec{x} \\
V \text{ basis} & V^{-1}\vec{x} \arrow[u, shift right, "V"'] \arrow[rr, "\Lambda"] & & \Lambda V^{-1}\vec{x} \arrow[u, "V"']
\end{tikzcd}$$

1. Coordinate Change of Basis

(a) *Transformation From Standard Basis To Another Basis in* $\mathbb{R}^3$
Calculate the coordinate transformation between the following bases:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$ where $\vec{x}_u$ contains the coordinates of a vector in a basis of the columns of $\mathbf{U}$ and $\vec{x}_v$ is the coordinates of the same vector in the basis of the columns of $\mathbf{V}$.

Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_v$. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now let $\vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. What is $\vec{x}_v$?

$\vec{x}_u$: coordinates in U basis
$\vec{x}_v$: - — — — V basis

U: Transformation from U basis to standard basis
V: Tran — . V basis standard basis.

$\vec{x} = U\vec{x}_u = V\vec{x}_v$

Goal: Find T : Cords in $U \longrightarrow$ Cords in V

$\vec{x}_v = T\vec{x}_u$

$\vec{x}_v = \underbrace{V^{-1}U}_{T}\vec{x}_u$

$$T = V^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \vec{x}_v = T\vec{x}_u = V^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$$\vec{x}_w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{x}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_w = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{x}_v = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

(b) *Transformation Between Two Bases in* $\mathbb{R}^3$

Calculate the coordinate transformation between the following bases:

$$\mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{W} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_w = \mathbf{T}\vec{x}_v$. Let $\vec{x}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_w$. Repeat this for $\vec{x}_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Now let $\vec{x}_v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{x}_w$?

Same as before.

$$\vec{x} = V\vec{x}_v = W\vec{x}_w$$

$$\vec{x}_w = \underbrace{W^{-1}V}_{T}\vec{x}_v$$

$$T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix}$$

Coords in V $\xrightarrow{V}$ Coords in standard basis

$\xrightarrow{W^{-1}}$ Coords in W

$$\vec{x}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{x}_w = T\vec{x}_v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{x}_w = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \vec{x}_w = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

2. Diagonalization square matrices $k \times k$

(a) Consider a matrix **A**, a matrix **V** whose columns are the eigenvectors of **A**, and a diagonal matrix **Λ** with the eigenvalues of **A** on the diagonal (in the same order as the eigenvectors (or columns) of **V**). From these definitions, show that

$$\mathbf{AV} = \mathbf{V\Lambda}$$

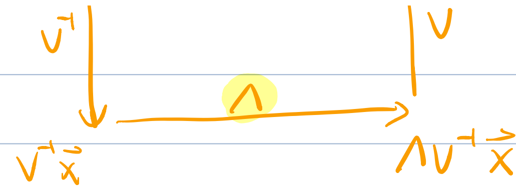
Let $\vec{v}_i$ be an eigenvec of A with eigenval λ_i

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

$$\begin{aligned} AV &= A[\vec{v}_1 \ \ \vec{v}_2 \ \cdots \ \vec{v}_k] \\ &= [A\vec{v}_1 \ \ A\vec{v}_2 \ \cdots \ A\vec{v}_k] \\ &= [\lambda_1\vec{v}_1 \ \ \lambda_2\vec{v}_2 \ \cdots \ \lambda_k\vec{v}_k] \\ &= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \ \lambda_k \end{bmatrix} \\ &= V\Lambda \end{aligned}$$

Diagonalization: $AV = V\Lambda \Rightarrow A = V\Lambda V^{-1}$

$$\vec{x} \xrightarrow{A} A\vec{x}$$



3. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage changes as a function of the derivative of the current across it. That is:

$$V_L(t) = L\frac{dI_L(t)}{dt}$$

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the equivalent "fundamental" circuit for an inductor:

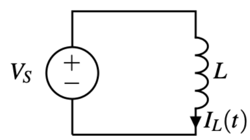


Figure 1: Inductor in series with a voltage source.

Henry

(a) What is the current through an inductor as a function of time? If the inductance is $L = 3\text{H}$, what is the current at $t = 6\text{s}$? Assume that the voltage source turns from 0V to 5V at time $t = 0\text{s}$, and there's no current flowing in the circuit before the voltage source turns on.

$I(0) = 0$

$$V_L(t) = L\frac{dI_L}{dt}$$

$$\frac{dI_L}{dt} = \frac{V_L(t)}{L} = \frac{V_S}{L}$$

slope is constant

$$I_L(t) = \frac{V_S}{L}t + \underbrace{I_L(0)}_{}$$

≈ 0

$$I_L(6) = \frac{5V}{3H}\cdot 6s = 10A$$

(b) Now, we add some resistance in series with the inductor, as in Figure 2.

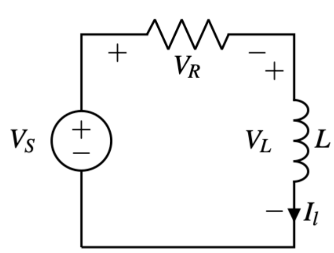


Figure 2: Inductor in series with a voltage source.

Solve for the current $I_L(t)$ in the circuit over time, in terms of R, L, V_S, t.

Use KVL $\quad V_s = \underbrace{V_R(t)}_{RI} + \underbrace{V_L(t)}_{L\frac{dI(t)}{dt}}$

$$\frac{dI(t)}{dt} = -\frac{R}{L}I(t) + \frac{V_s}{L}$$

solve for

$$I(t) = \frac{V_s}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

(c) **(Practice)** Suppose $R = 500\Omega, L = 1\text{mH}, V_S = 5\text{V}$. Plot the current through and voltage across the inductor $(I_L(t), V_L(t))$, as these quantities evolve over time.

4. Fibonacci Sequence

(a) The Fibonacci sequence is built as follows: the n-th number (F_n) is sum of the previous two numbers in the sequence. That is:

$$F_n = F_{n-1} + F_{n-2}$$

If the sequence is initialized with $F_1 = 0$ and $F_2 = 1$, then the first 11 numbers in the Fibonacci sequence are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

We can express this computation as a matrix multiplication:

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

What is $\mathbf{A}$?

$$F_n = 1 \cdot F_{n-1} + 1 \cdot F_{n-2}$$

$$F_{n-1} = 1 \cdot F_{n-1} + 0 \cdot F_{n-2}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

This should be F_{n-2}

(b) Find the eigenvalues and corresponding eigenvectors of **A**.

(c) Diagonalize **A** (that is, in the expression $\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$, solve for each component matrix.)

(d) Use the diagonalized result to show that we can arrive at an analytical result for any F_n:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$$

Why diagonalization:

$$A^k = (V \Lambda V^{-1})^k$$

$$= (V \Lambda V^{-1})(V \Lambda V^{-1}) \cdots (V \Lambda V^{-1})$$

$$= V \underbrace{\Lambda^k}_{\text{diagonal}} V^{-1}$$