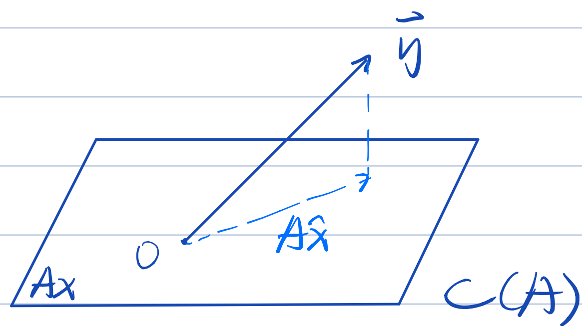
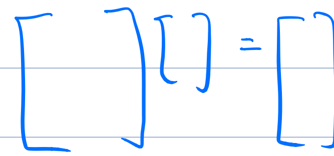
EECS 16B   DIS 7B   Gaoyue

· least Squares

$A\vec{x} = \vec{y}$

$\hat{x} = (A^TA)^{-1}A^T\vec{y}$





- Stability   BIBO

A system is stable if x(i) remains bounded for any initial condition and any bounded sequence of inputs u(i)

Continuous: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$

$Re\{\lambda_i(A)\} < 0$

Discrete: $x(t+1) = ax(t) + bu(t)$

1. System identification by means of least squares

Working through this question will help you understand better how we can use experimental data taken from a (presumably) linear system to learn a discrete-time linear model for it using the least-squares techniques you learned in 16A. You will later do this in lab for your robot car.

As you were told in 16A, least-squares and its variants are not just the basic workhorses of machine learning in practice, they are play a conceptually central place in our understanding of machine learning well beyond least-squares.

Throughout this question, you should consider measurements to have been taken from one long trace through time.

(a) Consider the scalar discrete-time system

noise

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{1}$$

Where the scalar state at time i is $x[i]$, the input applied at time i is $u[i]$ and $w[i]$ represents some external disturbance that also participated at time i.

Assume that you have measurements for the states $x[i]$ from $i = 0$ to m and also measurements for the controls $u[i]$ from $i = 0$ to $m-1$.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a and b.

Given: $x[0], x[1], \dots, x[m-1], x[m]$

$u[0], u[1], \dots, u[m-1]$

System: $x[i+1] = ax[i] + bu[i] + w[i]$

m eqs
$$\begin{cases} x[1] = ax[0] + bu[0] + w[0] \\ x[2] = ax[1] + bu[1] + w[1] \\ \vdots \\ x[m] = ax[m-1] + bu[m-1] + w[m-1] \end{cases}$$

Least Squares

$$\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[m] \end{bmatrix} = \begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \\ x[m-1] & u[m-1] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\vec{S} = D\vec{p}$$

(b) What if there were now two distinct scalar inputs to a scalar system

$$x[i+1] = ax[i] + b_1u_1[i] + b_2u_2[i] + w[i] \quad (2)$$

and that we have measurements as before, but now also for both of the control inputs.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a, b_1, b_2.

Given: $x[0], x[1], \ldots, x[m-1], x[m]$

$u_1[0], u_1[1], \ldots, u_1[m-1]$

$u_2[0], u_2[1], \ldots, u_2[m-1]$

Least Squares

$u_1 = \alpha u_2$

$$\underbrace{\begin{bmatrix} x[1] \\ \vdots \\ x[m] \end{bmatrix}}_{\vec{S}} = \begin{bmatrix} x[0] & u_1[0] & u_2[0] \\ \vdots & & \\ x[m-1] & u_1[m-1] & u_2[m-1] \end{bmatrix} \begin{bmatrix} a \\ b_1 \\ b_2 \end{bmatrix}$$

$$\vec{S} = D\vec{p}$$

(c) What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?

(d) Returning to the scalar case with only one input, what could go wrong? When would you be unable to use least-squares to get the parameters you want?

Least Squares Soln: $\vec{p} = (D^TD)^{-1} D^T \vec{s}$

D^TD may not be invertible.

D has linearly dependent columns

ex. c) inputs u_1, u_2 too similar

$\vec{u}_1 = \alpha \vec{u}_2$ $\qquad x[1] = 1$

d) $\vec{x} = \alpha \vec{u}$ $\qquad x[2] = 1$

$\boxed{(a = 0,\ b = 1,\ u = 1)}$ $\qquad x[3] = 1$

$x[i+1] = \underbrace{u[i]}_{1}$ $\qquad \vdots$

(e) Now consider the two dimensional state case with a single input.

$$\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \overset{\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}}{\vec{x}[i]} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \vec{w}[i] \tag{3}$$

How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$? What work/computation can we reuse across the two problems?

$x_1[i+1] = a_{11} x_1[i] + a_{12} x_2[i] + b_1 u[i] + w_1[i]$

$x_2[i+1] = a_{21} x_2[i] + a_{22} x_2[i] + b_2 u[i] + w_2[i]$

$$x_1[i+1] = \begin{bmatrix} x_1[i] & x_2[i] & u_1[i] \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \end{bmatrix} + w_1[i]$$

Given: $x_1[0], \ldots, x_1[m-1], x_1[m]$

$x_2[0], \ldots, x_2[m-1], x_2[m]$

$u[0], \ldots, u[m-1]$

For x_1:

$$\begin{bmatrix} x_1[1] \\ \vdots \\ x_1[m] \end{bmatrix} = \begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ \vdots & & \\ x_1[m-1] & x_2[m-1] & u[m-1] \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \end{bmatrix}$$

P_1 $\quad \vec{P_1}$

x_2:

$$\begin{bmatrix} x_2[1] \\ \vdots \\ x_2[m] \end{bmatrix} = \begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ \vdots & & \\ x_1[m-1] & x_2[m-1] & u[m-1] \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \\ b_2 \end{bmatrix}$$

$\vec{P_2}$

Can stack them:

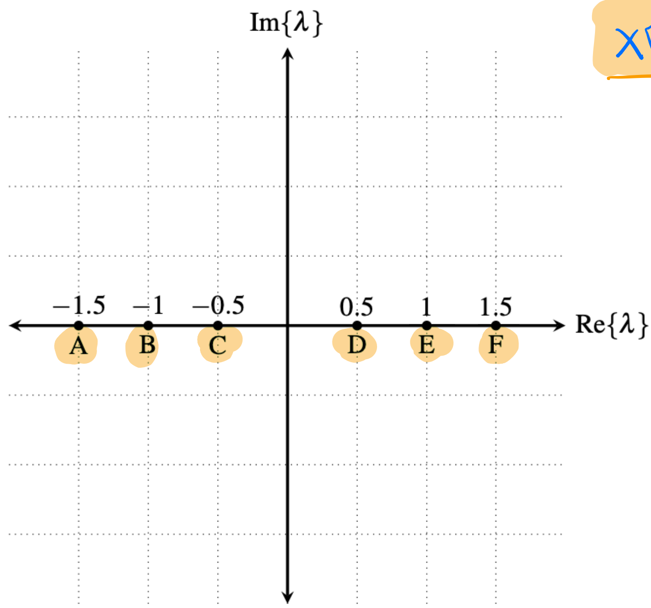
$$\begin{bmatrix} x_1[1] & x_2[1] \\ \vdots & \vdots \\ x_1[m] & x_2[m] \end{bmatrix} = \begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ \vdots & & \\ x_1[m-1] & x_2[m-1] & u[m-1] \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ b_1 & b_2 \end{bmatrix}$$

$S = D P$

2. Discrete time system responses

We have a system $\boxed{x[k+1] = \lambda x[k]}$. For each λ value plotted on the real-imaginary axis, sketch $x[k]$ with an initial condition of $x[0] = 1$. Determine if each system is stable.

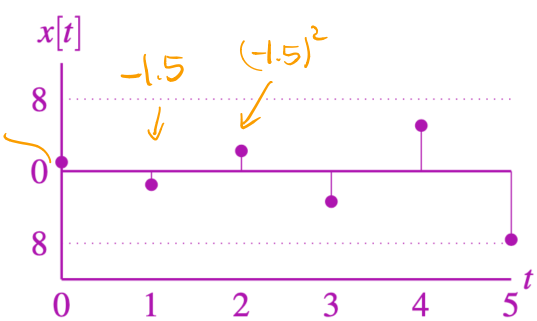
$x[k] = \lambda^k x[0]$



System A, $\lambda = -1.5$

unstable.

$x[0]=1$

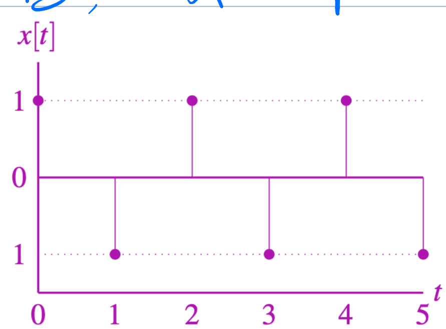


$x[1] = -1.5 \cdot 1$

$x[2] = (-1.5)^2 \cdot 1$

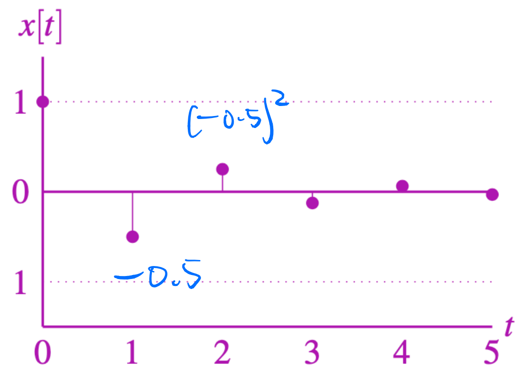
$x[3] = (-1.5)^3 \cdot 1$

System B, $\lambda = -1$



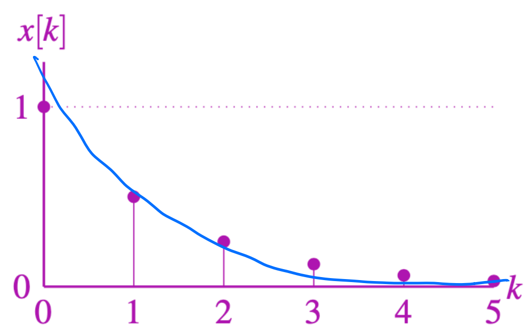
Marginally stable.

System C, $\lambda = -0.5$



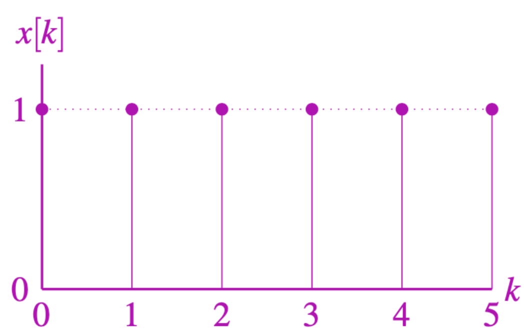
Stable

System D, $\lambda = 0.5$



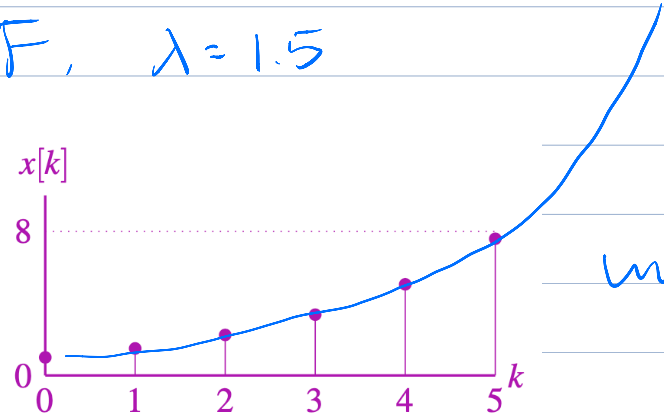
Stable

System E, $\lambda = 1$

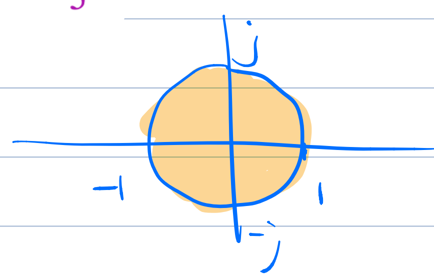


Marginally stable

System F, $\lambda = 1.5$

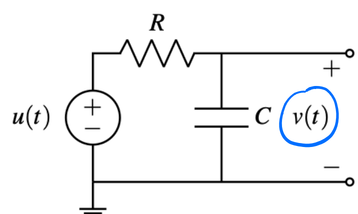


unstable.



3. Stability Examples and Counterexamples

(a) Consider the circuit below with $R = 1\Omega$, $C = 0.5\text{F}$, and $u(t) = \cos(t)$. Furthermore assume that $v(0) = 0$ (that the capacitor is initially discharged).



This circuit can be modeled by the differential equation

$$\frac{d}{dt}v(t) = -2v(t) + 2u(t) \tag{4}$$

Show that the differential equation is always stable. Consider what this means in the physical circuit.

Hint: try letting $-2v(t) + 2u(t) > 0$

$u(t) > v(t)$

$-1 < u(t) < 1$

(b) Consider the discrete system

$$x[k+1] = 2x[k] + u[k] \tag{5}$$

with $x[0] = 0$.

Is the system stable or unstable? If unstable, find a bounded input sequence $u[k]$ that causes the system to "blow up". If unstable, is there still a (non-trivial) bounded input sequence that does not cause the system to "blow up"?

Unstable!

blow up: $u[k] = 1, \underline{0, 0, \ldots}$

$1, 2, 4, 8, \ldots$

Stable output: $u[k] = 1, -2, 0, \ldots$

$x[0] = 0$

$2x[k] + u[k] = 1, 0, 0, 0 \ldots$