1. DFT

In order to get practice with calculating the Discrete Fourier Transform (DFT), this problem will have you calculate the DFT for a few variations on a cosine signal.

Consider a sampled signal that is a function of discrete time $x[t]$. We can represent it as a vector of discrete samples over time $\vec{x}$, of length N.

$$\vec{x} = \begin{bmatrix} x[0] & \dots & x[N-1] \end{bmatrix}^T \tag{1}$$

Let $\vec{X} = \begin{bmatrix} X[0] & \dots & X[N-1] \end{bmatrix}^T$ be the signal $\vec{x}$ represented in the frequency domain, then

$$\vec{x} = U\vec{X} \tag{2}$$

and the inverse operation is given by

$$\vec{X} = U^{-1}\vec{x} = U^*\vec{x} \tag{3}$$

where the columns of U are the orthonormal DFT basis vectors.

$$U = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & e^{j\frac{2\pi(2)}{N}} & \cdots & e^{j\frac{2\pi(N-1)}{N}} \\ 1 & e^{j\frac{2\pi(2)}{N}} & e^{j\frac{2\pi(4)}{N}} & \cdots & e^{j\frac{2\pi 2(N-1)}{N}} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{j\frac{2\pi(N-1)}{N}} & e^{j\frac{2\pi 2(N-1)}{N}} & \cdots & e^{j\frac{2\pi(N-1)(N-1)}{N}} \end{bmatrix} \tag{4}$$

$$= \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^1 & \omega_N^2 & \cdots & \omega_N^{(N-1)} \\ 1 & \omega_N^2 & \omega_N^{2\cdot 2} & \cdots & \omega_N^{(N-1)2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)(N-1)} \end{bmatrix}, \tag{5}$$

where $\omega_N = e^{j\frac{2\pi}{N}}$ is the Nth primitive root of unity.

We ~~sometimes call~~ the components of $\vec{X}$ the *DFT coefficients* of the time-domain signal $\vec{x}$. We can think of the components of $\vec{X}$ as weights that represent $\vec{x}$ in the DFT basis.

(a) Let's begin by looking at the DFT of $x_1[n] = \cos\left(\frac{2\pi}{5}n\right)$ for $N = 5$ samples $n \in \{0, 1, \ldots, 4\}$. Compute the DFT basis matrix U.

$$N = 5 \Rightarrow W_N = e^{j\frac{2\pi}{5}}$$

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{5}} & e^{j\frac{4\pi}{5}} & e^{j\frac{6\pi}{5}} & e^{j\frac{8\pi}{5}} \\ 1 & e^{j\frac{4\pi}{5}} & e^{j\frac{8\pi}{5}} & e^{j\frac{12\pi}{5}} & e^{j\frac{16\pi}{5}} \\ 1 & e^{j\frac{6\pi}{5}} & e^{j\frac{12\pi}{5}} & e^{j\frac{18\pi}{5}} & e^{j\frac{24\pi}{5}} \\ 1 & e^{j\frac{8\pi}{5}} & e^{j\frac{16\pi}{5}} & e^{j\frac{24\pi}{5}} & e^{j\frac{32\pi}{5}} \end{bmatrix}$$

$$\vec{u}_i[n] = \frac{1}{\sqrt{5}} e^{j\frac{(2i)\pi}{5}n}$$

(b) Write out $\vec{x}_1$ in terms of the DFT basis vectors.

$$x_1[n] = \cos\left(\frac{2\pi}{5}n\right) = \frac{1}{2}\left(e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n}\right)$$

$$= \frac{1}{2}\left(e^{j\frac{2\pi}{5}n} + e^{j\frac{8\pi}{5}n}\right)$$

$$= \frac{\sqrt{5}}{2}\left(\frac{1}{\sqrt{5}}e^{j\frac{2\pi}{5}n} + \frac{1}{\sqrt{5}}e^{j\frac{8\pi}{n}}\right)$$

$$= \frac{\sqrt{5}}{2}\left(\vec{u}_1[n] + \vec{u}_4[n]\right)$$

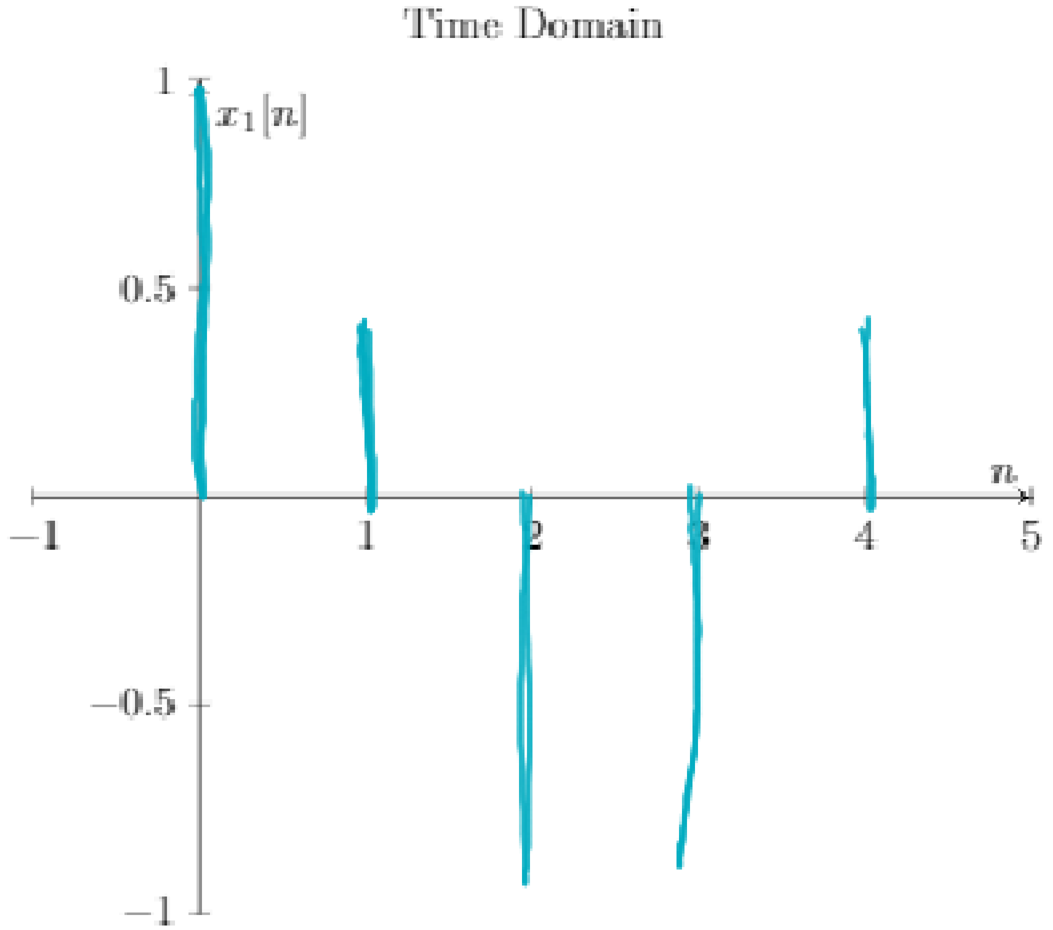
$$\vec{x}_1 = \frac{\sqrt{5}}{2}\vec{u}_1 + \frac{\sqrt{5}}{2}\vec{u}_4$$

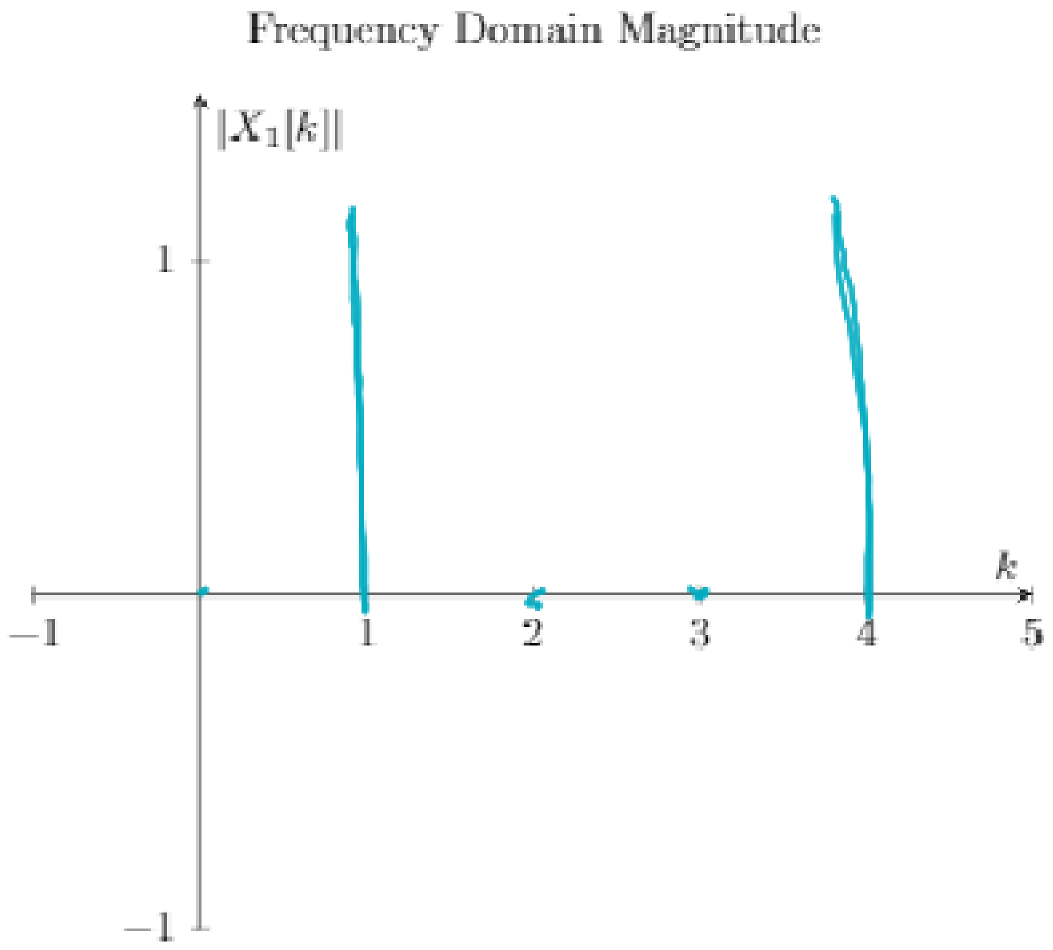
(c) Find the DFT coefficients $X_1[k]$.

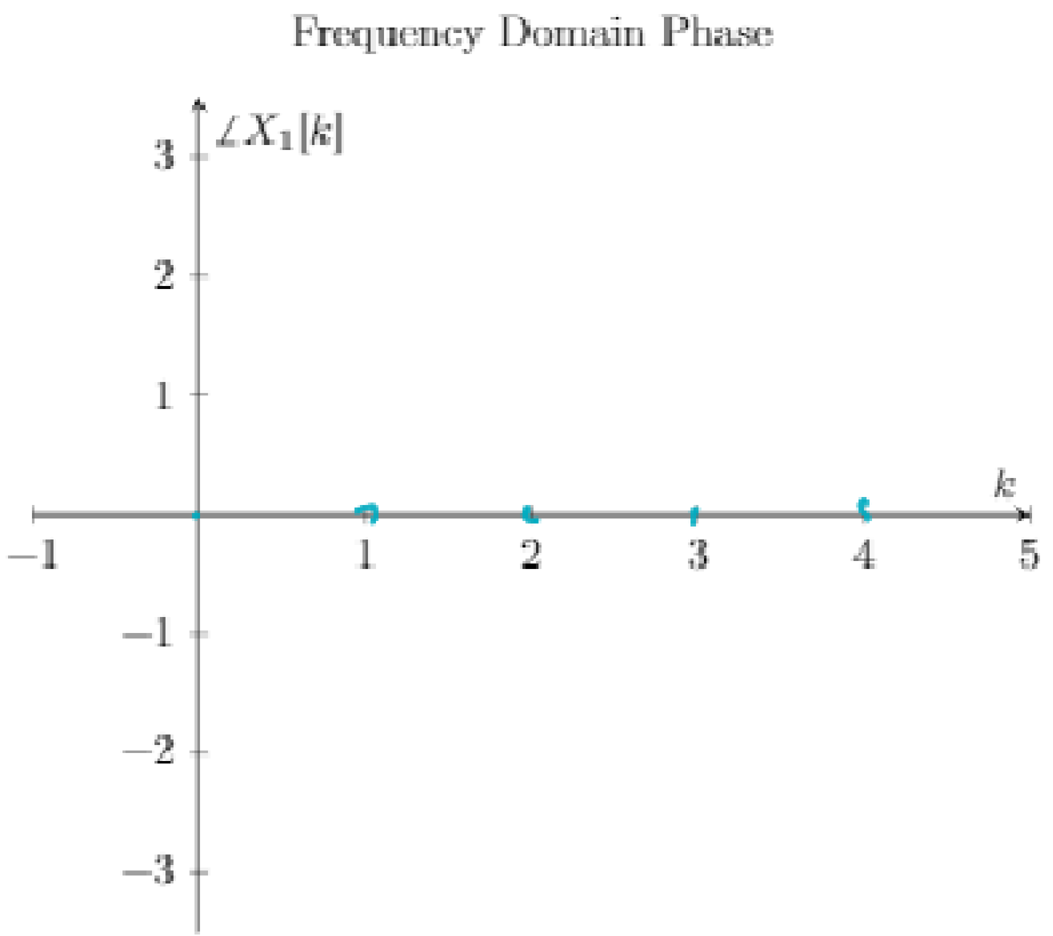
$$\vec{X}_1 = U^T \vec{x}_1 = \begin{bmatrix} - \vec{u}_0^* - \\ - \vec{u}_1^* - \\ - \vec{u}_2^* - \\ - \vec{u}_3^* - \\ - \vec{u}_4^* - \end{bmatrix} \left(\frac{\sqrt{5}}{2} \vec{u}_1 + \frac{\sqrt{5}}{2} u_4 \right)$$

$$= \begin{bmatrix} 0 \\ \frac{\sqrt{5}}{2} \\ 0 \\ 0 \\ \frac{\sqrt{5}}{2} \end{bmatrix}$$

(d) Plot the time domain representation of $x_1[n]$. Plot the magnitude, $|X_1[k]|$, and plot the phase, $\angle X_1[k]$, for the DFT representation $\vec{X}_1$.







(e) Now let's consider the case were have a non-zero phase. Let $x_2[n] = \cos\left(\frac{4\pi}{5}n + \pi\right)$. Find the DFT coefficients $\vec{X}_2$ for $\vec{x}_2$.

$$\cos\left(\frac{4\pi}{5}n + \pi\right) = \frac{1}{2}\left(e^{j\frac{4\pi}{5}}e^{j\pi} + e^{-j\frac{4\pi}{5}}e^{-j\pi}\right)$$

$$= \frac{1}{2}\left(e^{j\frac{4\pi}{5}}e^{j\pi} + e^{\frac{6\pi}{5}}e^{-j\pi}\right)$$

$$= \frac{\sqrt{5}}{2}\left(\frac{1}{\sqrt{5}}e^{j\frac{4\pi}{5}}e^{j\pi} + \frac{1}{\sqrt{5}}e^{\frac{6\pi}{5}}e^{-j\pi}\right)$$

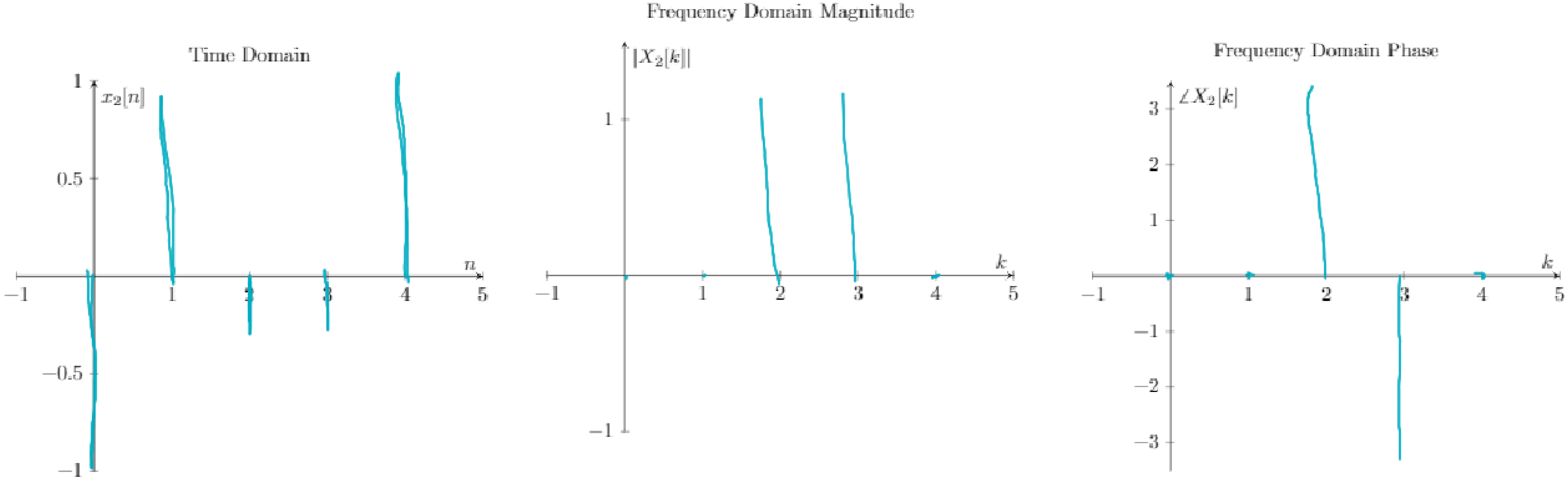
$$= \frac{\sqrt{5}}{2}\left(u_2[n]e^{j\pi} + u_3[n]e^{-j\pi}\right)$$

$$= \frac{\sqrt{5}}{2}e^{j\pi}u_2[n] + \frac{\sqrt{5}}{2}e^{-j\pi}u_3[n]$$

$$\vec{x} = \frac{\sqrt{5}}{2}e^{j\pi}\vec{u}_2 + \frac{\sqrt{5}}{2}e^{-j\pi}\vec{u}_3$$

$$\vec{X} = U^*\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \frac{\sqrt{5}}{2}e^{j\pi} \\ \frac{\sqrt{5}}{2}e^{-j\pi} \\ 0 \end{bmatrix}$$

(f) Plot the time domain representation of $x_2[n]$. Plot the magnitude, $|X_2[k]|$, and plot the phase, $\angle X_2[k]$, for the DFT representation $\vec{X}_2$.



(g) Now let's look at the reverse direction. Given $\vec{X}_3 = \begin{bmatrix} 2 & e^{-j\frac{\pi}{2}} & 0 & 0 & e^{j\frac{\pi}{2}} \end{bmatrix}^\top$, find $x_3[n]$.

$$\vec{x}_3 = U\vec{X}_3 = [u_0 \ \cdots \ u_n] \begin{bmatrix} 2 \\ e^{-j\frac{\pi}{2}} \\ 0 \\ 0 \\ e^{j\frac{\pi}{2}} \end{bmatrix}$$

$$= 2\vec{u}_0 + e^{-j\frac{\pi}{2}}\vec{u}_1 + e^{j\frac{\pi}{2}}\vec{u}_4$$

$$\frac{8\pi}{5} - \frac{10\pi}{5} = \frac{-2\pi}{5}$$

$$\vec{x}_3[n] = \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} e^{-j\frac{\pi}{2}} e^{j\frac{2\pi}{5}n} + \frac{1}{\sqrt{5}} e^{j\frac{\pi}{2}} e^{-j\frac{2\pi}{5}n}$$

$$= \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}}\frac{1}{2}\left(e^{j\frac{2\pi n}{5}} e^{-j\frac{\pi}{2}} + e^{-j\frac{2\pi n}{5}} e^{j\frac{\pi}{2}}\right)$$

$$= \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} \cos\left(\frac{2\pi}{5}n - \frac{\pi}{2}\right)$$