1. Eigenvalues Placement in Discrete Time

Consider the following linear discrete time system

$$\vec{x}[t+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}}_{A} \vec{x}[t] + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b} u[t] + \vec{w}[t] \tag{1}$$

(a) Is the system given in eq. (1) stable?

$$\det(A - \lambda I) = \left| \begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} \right| = (-\lambda)(-1-\lambda) - 2$$

$$\Rightarrow \lambda^2 + \lambda - 2 = 0 \Rightarrow (\lambda - 1)(\lambda + 2) = 0$$

$$\lambda = 1, -2$$

(b) Derive a state space representation of the resulting closed loop system using state feedback of the form $u[t] = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[t]$.

Hint: If you're having trouble parsing this expression for $u[t]$, note that $\begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is a *row vector*, while $\vec{x}[t]$ is a *column vector*. What happens when we multiply a row vector with a column vector like this?

$$\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix}$$

$$\begin{aligned}
\vec{x}[t+1] &= A\vec{x}[t] + \vec{b}u[t] \\
&= A\vec{x}[t] + \vec{b}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\vec{x}[t] \\
&= \left(A + \vec{b}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\vec{x}[t] \\
&= \underbrace{\left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)}\vec{x}[t] = \underbrace{\begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}}_{\tilde{A}}\vec{x}[t]
\end{aligned}$$

(c) Find the appropriate state feedback constants, k_1, k_2, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

$$\left| \begin{bmatrix} k_1 - \lambda & 1 + k_2 \\ 2 & -1-\lambda \end{bmatrix} \right| = (k_1 - \lambda)(-1-\lambda) - 2(1+k_2)$$

$$= \lambda^2 + \underbrace{(1-k_1)}\lambda + \underbrace{(-k_1 - 2k_2 - 2)}$$

$$\left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) = \lambda^2 - \frac{1}{4}$$

$$1 - k_1 = 0 \Rightarrow k_1 = 1$$

$$-k_1 - 2k_2 - 2 = -\frac{1}{4}$$

$$-1 - 2k_2 - 2 = -\frac{1}{4} \Rightarrow k_2 = \frac{-11}{8}$$

(d) Is the system now stable?

$$\left|\frac{1}{2}\right| < 1$$

$$\left|\frac{-1}{2}\right| < 1$$

(e) Suppose that instead of $\overset{b}{\begin{bmatrix}1\\0\end{bmatrix}}u[t]$ in eq. (1), we had $\begin{bmatrix}1\\1\end{bmatrix}u[t]$ as the way that the discrete-time control acted on the system. As before, we use $u[t] = \begin{bmatrix}k_1 & k_2\end{bmatrix}\vec{x}[t]$ to try and control the system. What would the eigenvalues be? Can you move all the eigenvalues to where you want? Give an intuitive explanation of what is going on.

$$\vec{x}[t+1] = \left(\begin{bmatrix}0 & 1\\2 & -1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}\right)\vec{x}[t]$$

$$\begin{bmatrix}k_1 & k_2+1\\2+k_1 & k_2-1\end{bmatrix}$$

$$\left|\begin{bmatrix}k_1-\lambda & k_2+1\\2+k_1 & k_2-1-\lambda\end{bmatrix}\right| = (k_1-\lambda)(k_2-1-\lambda)-(k_1+2)(k_2+1)$$

$$= k_1(k_2-1)-k_1\lambda-\lambda(k_2-1)+\lambda^2-(k_1k_2+k_1+2k_2+2)$$

$$= k_1k_2-k_1-k_1\lambda-\lambda k_2+\lambda+\lambda^2-k_1k_2-k_1-2k_2-2$$

$$= \lambda^2+(1-k_1-k_2)\lambda-2(1+k_1+k_2)$$

$$= \underbrace{(\lambda+2)}\,\underbrace{(\lambda-(1+k_1+k_2))} \qquad \lambda=-2$$

2. Controlling states by designing sequences of inputs

This is something that you saw in 16A in the Segway problem. In that problem, you were given a semi-realistic model for a segway. Here, we are just going to consider the following matrix chosen for ease of understanding what is going on:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let's assume we have a *discrete-time* system defined as follows:

$$\vec{x}[t+1] = A\vec{x}[t] + \vec{b}u[t].$$

(a) We are given the initial condition $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let's say we want to achieve $\vec{x}[T] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ for some specific $T \geq 0$. We don't need to stay there, we just want to be in this state at that time. What is the smallest T such that this is possible? What is our choice of sequence of inputs $u[t]$?

$$\vec{X}[0] = \begin{bmatrix} x_1[0] \\ x_2[0] \\ x_3[0] \\ x_4[0] \end{bmatrix} \qquad \vec{X}[1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \\ x_3[0] \\ x_4[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} U[0]$$

$$U[0] = U[0] \qquad = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ U[0] \end{bmatrix}$$

$$= \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ U[0] \end{bmatrix}$$

$$\vec{X}[2] = \begin{bmatrix} x_3[0] \\ x_4[0] \\ u[0] \\ u[1] \end{bmatrix}, \quad \vec{x}[3] = \begin{bmatrix} x_4[0] \\ u[0] \\ u[1] \\ u[2] \end{bmatrix}$$

$$\vec{X}[4] = \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix}$$

$T = 4$

$u[0] = 1$

$u[1] = 2$

$u[2] = 3$

$u[3] = 4$

(b) What if we started from $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest T and what is our choice of $u[t]$?

$u[0] = 4$

$\vec{x}[1] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(c) What if we started from $\vec{x}[0] = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$? What is the smallest T and what is our choice of $u[t]$?

$T = 4$

$\vec{x}[4] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$u[0] = 1$

$u[1] = 2$

$u[2] = 3$

$u[3] = 4$

3. Uncontrollability

Consider the following discrete-time system with the given initial state:

$$\vec{x}[t+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[t]$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) Is the system controllable?

$$C = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

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rank = 2

(b) Is it possible to reach $\vec{x}[T] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some $t = T$? For what input sequence $u[t]$ up to $t = T - 1$?

$$X[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}[1] = A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \vec{b}\,u[0] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[0] = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2u[0] \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix}$$

$$\vec{x}[2] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[1] = \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2u[1] \end{bmatrix} = \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix}$$

$$\vec{x}[t] = \begin{bmatrix} 2^t x_1[0] \\ -3x_1[t-1] + x_3[t-1] \\ x_2[t-1] + 2u[t-1] \end{bmatrix}$$

(c) Is it possible to reach $\vec{x}[T] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some $t = T$? For what input sequence $u[t]$ up to $t = T - 1$?

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

$$u[0] = -1$$

(d) Find the set of all possible states reachable after two timesteps.

$$\vec{x}[2] = \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$