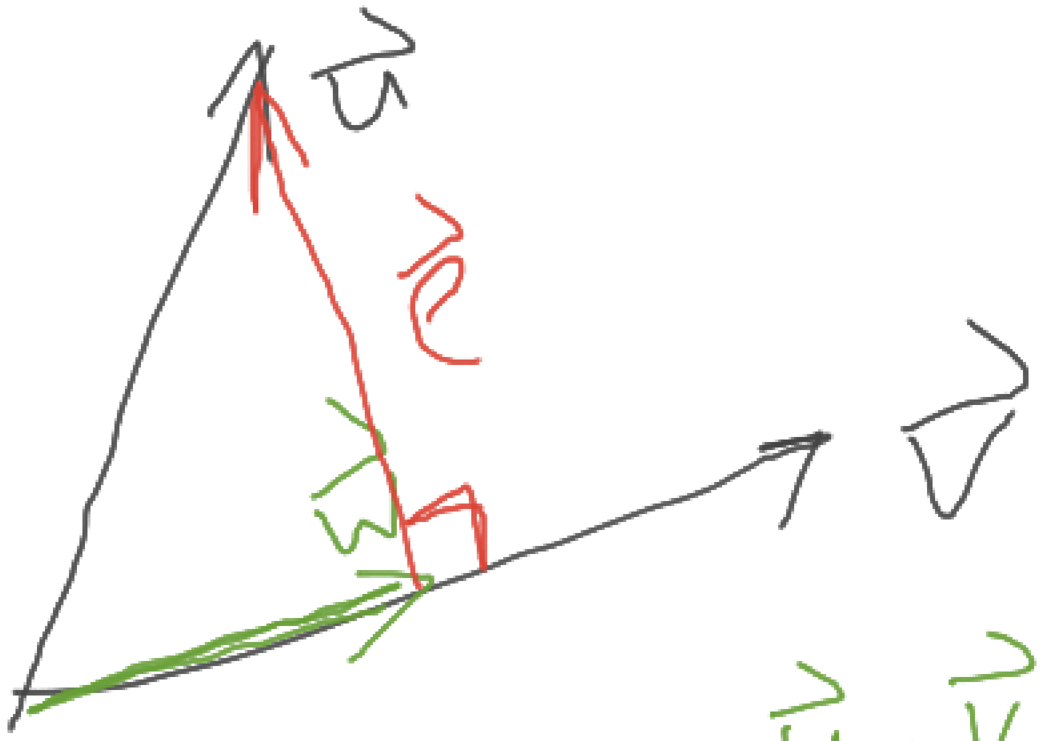

$$\operatorname{proj}_{\vec{v}} \vec{u} = \vec{w} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \vec{v}$$

$$\vec{e} = \vec{u} - \vec{w} = \vec{u} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \vec{v}$$

1. Gram-Schmidt Algorithm

Let's apply Gram-Schmidt orthonormalization to a set of three linearly independent vectors $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$.

(a) Find unit vector $\vec{q}_1$ such that $\text{span}(\{\vec{q}_1\}) = \text{span}(\{\vec{s}_1\})$.

$$\vec{q}_1 = \frac{\vec{s}_1}{\|\vec{s}_1\|}$$

(b) Given $\vec{q}_1$ from the previous step, find $\vec{q}_2$ such that $\text{span}(\{\vec{q}_1, \vec{q}_2\}) = \text{span}(\{\vec{s}_1, \vec{s}_2\})$ and $\vec{q}_2$ is orthogonal to $\vec{q}_1$.

What would happen if $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ were *not* linearly independent, but rather $\vec{s}_1$ were a multiple of $\vec{s}_2$?

$$\vec{q}_2 = \vec{s}_2 - \text{proj}_{\vec{q}_1} \vec{s}_2 = \frac{\vec{s}_2 - \frac{\vec{s}_2 \cdot \vec{q}_1}{\|\vec{q}_1\|} \vec{q}_1}{\left\| \vec{s}_2 - \frac{\vec{s}_2 \cdot \vec{q}_1}{\|\vec{q}_1\|} q_1 \right\|}$$

(c) Now given $\vec{q}_1$ and $\vec{q}_2$ in the previous steps, find $\vec{q}_3$ such that $\text{span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) = \text{span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$, and $\vec{q}_3$ is orthogonal to both $\vec{q}_1$ and $\vec{q}_2$, and finally $\|\vec{q}_3\| = 1$.

$$\text{proj}_{\vec{q}_1, \vec{q}_2} \vec{s}_3 = \frac{\vec{s}_3 \cdot \vec{q}_1}{\|\vec{q}_1\|} \vec{q}_1 + \frac{\vec{s}_3 \cdot \vec{q}_2}{\|\vec{q}_2\|} \vec{q}_2 = \vec{w}_3$$

$$\vec{z}_3 = \vec{s}_3 - \vec{w}_3$$

$$\vec{q}_3 = \frac{\vec{z}_3}{\|z_3\|}$$

(d) Let's extend this algorithm to n linearly independent vectors. That is, given an input $\{\vec{s}_1, \ldots, \vec{s}_n\}$, write the algorithm to calculate the orthonormal set of vectors $\{\vec{q}_1, \ldots, \vec{q}_n\}$, where $\text{span}(\{\vec{s}_1, \ldots, \vec{s}_n\}) = \text{span}(\{\vec{q}_1, \ldots, \vec{q}_n\})$.

Hint: How would you calculate the i^{th} vector, $\vec{q}_i$?

1) $s_1 \rightarrow$ normalize $\quad \vec{q}_1 = \dfrac{\vec{s}_1}{\|s_1\|}$

for $i = 2$ to n

$\vec{s}_i$ project onto $\vec{q}_1 - q_{i-1} \longrightarrow \vec{w}_i$

$\vec{z}_i = \vec{s}_i - \vec{w}_i$

$q_i = \dfrac{\vec{z}_i}{\|z_i\|}$

2. The Order of Gram-Schmidt

If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the set of vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Perform Gram-Schmidt on these vectors first in the order $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \vec{v}_1$$

$$\vec{q}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{q}_1}{\|\vec{q}_1\|}\vec{q}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{q}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{q}_1}{\|\vec{q}_1\|}\vec{q}_1 - \frac{\vec{v}_3 \cdot \vec{q}_2}{\|\vec{q}_2\|}\vec{q}_2 = \vec{v}_3 - \vec{q}_1 - q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(b) Now perform Gram-Schmidt on these vectors in the order $\vec{v}_3$, $\vec{v}_2$, $\vec{v}_1$. Do you get the same result?

$$\vec{q}_1 = \frac{\vec{V}_3}{\|v_3\|} = \frac{1}{\sqrt{3}}\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{3}}$$

$$\vec{z}_2 = \vec{V}_2 - \frac{\vec{V}_2 \cdot \vec{q}_1}{\|q_1\|}\vec{q}_1 = \vec{v}_2 - \frac{\vec{V}_2 \cdot \vec{V}_3}{\|\vec{V}_3\|}\cdot\frac{\vec{V}_3}{\|\vec{V}_3\|} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$\vec{q}_2 = \frac{\vec{z}_2}{\|z_2\|} = \frac{1}{\sqrt{\frac{2}{3}}}\vec{z}_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\sqrt{2/3} \end{bmatrix}$$

$$\vec{z}_3 = \vec{V}_1 - \frac{\vec{v}_1 \cdot \vec{v}_3}{\|v_3\|^2}\vec{v}_3 - \frac{\vec{v}_1 \cdot \vec{z}_2}{\|\vec{z}_2\|^2}\vec{z}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\vec{q}_3 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$