EECS 16B   Module 2/3 Lecture 7.

Announcements:
- MT scores released
- 1-1 conversations in OH. + appointments.

Today: An exploration in 2 parts.
Story completed in next lecture.

① Symmetric matrices $\longleftrightarrow$ Spectral theorem
↳ why are they special.

② The importance of finding the right basis.
- Differential eqn.
- Minimum energy control.

③ Is there a way to "always" find the right basis?
↳ Non-square, non-symmetric matrix $\rightarrow$ "right" basis.
↳ Enter *SVD*. (Singular Value Decomposition)

Last time: $M = UTU^{-1}$ T is upper tri.
"Schur form"   U is orthonormal.

What if we had a symmetric matrix? S

S is symmetric $S = S^T$

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} \qquad S_{ij} = S_{ji} \qquad \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

Upper-triangularize S.

$$S = UTU^{-1} = UTU^T$$ ← U is orthonormal

Taking transpose.

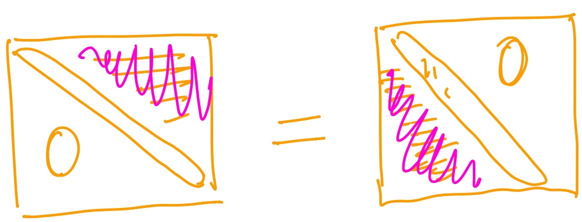
$$S^T = (UTU^T)^T = (U^T)^T (UT)^T$$

$$S^T = UT^TU^T$$

We know: $\underline{\underline{S = S^T}}$

$$\Rightarrow UTU^T = UT^TU^T$$

$$\Rightarrow T = T^T$$



$\Rightarrow$ All non-diagonal entries of T, must be 0!

$\Rightarrow$ T must be diagonal!

If S is a symmetric matrix, U, basis for upper-triangularization, actually diagonalizes S!

$$S = U T U^{-1}$$

$$S = U \begin{bmatrix} \ddots & \\ & \ddots \end{bmatrix} U^{-1}$$ "Diagonalizing S"

16A: If a matrix has distinct e-vals, it can be diagonalized $\rightarrow$ eigenvectors form a basis.

1. Symm matrix can be diagonalized. Always

Symmetric matrix: $A : (A^T A)$ is symmetric.

$S = U D U^T$ Because T is diagonal. $T = D$

$SU = U D U^T U$ $\boxed{U^T U = I}$

$SU = U \cdot D \cdot I = UD.$

$$S \begin{bmatrix} \vec{u_1} & \vec{u_2} & \cdots & \vec{u_n} \end{bmatrix} = \begin{bmatrix} \vec{u_1} & \cdots & \vec{u_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$\left.\begin{aligned} S \cdot \vec{u_1} &= \vec{u_1} \cdot \lambda_1 \\ S\, \vec{u_2} &= \vec{u_2} \cdot \lambda_2 \\ &\vdots \end{aligned}\right\}$$ $\vec{u_1}, \vec{u_2} \ldots \vec{u_n}$ must be the eigenvectors of S!

2. The diagonalizing basis / matrix U, is made up

of the eigenvectors of the matrix S!

Because U was orthonormal, $\Rightarrow$

$\vec{u_1}, \vec{u_2} \dots \vec{u_n}$ are orthonormal

$\Rightarrow$ eigenvectors of S are orthonormal!!!

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(3) Can you say anything about e-vals of S?

2x2:
$$S = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$$

$$(S - \lambda I) = \begin{bmatrix} \alpha-\lambda & \beta \\ \beta & \gamma-\lambda \end{bmatrix}.$$

$$\text{Det}(S - \lambda I) = (\alpha-\lambda)(\gamma-\lambda) - \beta^2$$
$$= \lambda^2 - (\alpha+\gamma)\cdot\lambda - \beta^2 + \alpha\gamma$$

"$ax^2 + bx + c$" quadratic

Discriminant : $b^2 - 4ac$

Discriminant: $(-(\alpha+\gamma))^2 - 4(1)(-\beta^2 + \alpha\gamma)$

$$= \alpha^2 + \gamma^2 + 2\alpha\gamma + 4\beta^2 - 4\alpha\gamma$$

$$= \alpha^2 + \gamma^2 - 2\alpha\gamma + 4\beta^2$$

$$= \underbrace{\underbrace{(\alpha-\gamma)^2}_{\geq 0} + \underbrace{4\beta^2}_{\geq 0}}_{\geq 0}$$

$$\geq 0$$

$\Rightarrow$ Quadratic equation must have real roots!

$\rightarrow$ λ's of S must be real numbers!

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③ Thm: If S is real symmetric, then it's eigenvalues must be real numbers

Proof:

Know: $S = S^T$

$S\vec{u} = \lambda \cdot \vec{u}$ — $(*)$.

λ is an eigenvalue, $\vec{u}$ is an eigenvector.

$\lambda = \alpha + \beta j$ $\qquad \bar{\lambda}_{conj} = \alpha - \beta j$

If possible, let us assume that λ is complex.

$\Rightarrow \lambda \neq \bar{\lambda}_{conj}$

Taking conjugates on both sides of (*)

$$\bar{S}\, \bar{\vec{v}}_{conj} = \bar{\lambda}_{conj}\, \bar{\vec{v}}_{conj}$$

real matrix

e.g. $S = \begin{bmatrix} a+bj & c+dj \\ e+fj & g+rj \end{bmatrix}$ $\qquad \bar{S} = \begin{bmatrix} a-bj & c-dj \\ e-fj & g-rj \end{bmatrix}$

If S is real

$$\bar{S} = S$$

$$\boxed{S\, \bar{\vec{v}}_{conj} = \bar{\lambda}_{conj}\, \bar{\vec{v}}_{conj}} \quad \text{— (I)}$$

Now, transpose (*).

$$(S\vec{v})^T = (\lambda \vec{v})^T$$

$$\Rightarrow \vec{v}^T S^T = \lambda \vec{v}^T$$

$$\Rightarrow \boxed{\vec{v}^{T} S = \lambda \vec{v}^{T}} \quad — (2)$$

Try to make these eq^ns similar:

(1) Left multiply by $\vec{v}^{T}$

$$(i) \Rightarrow \boxed{\vec{v}^{T} S \bar{\vec{v}}_{conj} = \vec{v}^{T} \bar{\lambda}_{conj} \cdot \bar{\vec{v}}_{conj}}$$

(2) Right multiply by $\bar{\vec{v}}_{conj}$ EQUAL

$$\boxed{\vec{v}^{T} S \cdot \bar{\vec{v}}_{conj} = \lambda \cdot \vec{v}^{T} \cdot \bar{\vec{v}}_{conj}}$$

$\Rightarrow$ We conclude:

$$\vec{v}^{T} \bar{\lambda}_{conj} \bar{\vec{v}}_{conj} = \lambda \vec{v}^{T} \bar{\vec{v}}_{conj}$$

$$\bar{\lambda}_{conj} \vec{v}^{T} \bar{\vec{v}}_{conj} = \lambda \vec{v}^{T} \bar{\vec{v}}_{conj} \quad — (3)$$

We have to check: is $\vec{v}^{T} \bar{\vec{v}}_{conj} = 0$??

$$\vec{v} = \begin{bmatrix} a+bj \\ c+dj \end{bmatrix} \qquad \vec{v}_{conj} = \begin{bmatrix} a-bj \\ c-dj \end{bmatrix}$$

$$\vec{v}^T \cdot \vec{v}_{conj} = (a+bj)(a-bj) + (c+dj)(c-dj)$$

$$= a^2+b^2 + c^2+d^2$$

Can this be zero?

$$\neq 0.$$

Because $\vec{v} \neq 0$. $\vec{v}$ is an eigenvector.

Hence, it is not the zero vector!

$\vec{v}^T \vec{v}_{conj}$ cannot $= 0$.

$\Rightarrow$ I can ~~[illegible]~~ cancel $\vec{v}^T \vec{v}_{conj}$ in eq (3)

$$\Rightarrow \quad \overline{\lambda}_{conj} = \lambda$$

$\Rightarrow$ λ must be real!!!

QED.

- Control
  - **Stability**: Blow up X
  - **Controllability**: Can I get where I want?
  - **Efficency**: What is the best path from A → B.

Control system:

$$\vec{x}[k+1] = A\vec{x}[k] + \vec{b}\,u[k]$$

$$\vec{x}[K] = A^K\vec{x}[0] + A^{k-1}\vec{b}\,u[0] + \cdots + \cdots + \vec{b}\,u[k-1]$$

Consider $k = 100$

$$\vec{x}[0] = 0$$

$$\vec{x}[100] = A^{99}\vec{b}\,u[0] + \cdots + \vec{b}\,u[99]$$

$$= \underbrace{\begin{bmatrix} \vec{b} & A\vec{b} & \cdots\cdots & A^{99}\vec{b} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} u[99] \\ u[98] \\ \vdots \\ \vdots \\ u[0] \end{bmatrix}}_{\vec{u}}$$

$$= C \cdot \vec{u}$$

$$\min \quad \|\vec{u}\|^2$$ "Minimum-energy control"

s.t.

$$\vec{x}[100] = C \cdot \vec{u}$$