EECS 16B : Module 3, Lecture 2

- More SVD : Heart of lots of ML.
- Principal Component Analysis (PCA)
  - → Neural analysis
  - → Netflix recommendations.

Reminder TI Design Contest

@1091 Piazza.

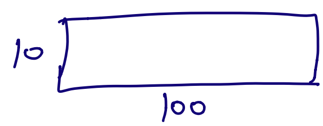
- $$$. Extra Pants.

u v

Reference: $A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 8 & 10 \end{bmatrix}_{2\times 4}$

$$A = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}}_{\substack{U \\ 2\times 2}} \underbrace{\begin{bmatrix} \sqrt{230} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\substack{\Sigma \\ 2\times 4}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{46}} & \sqrt{\frac{2}{23}} & 2\sqrt{\frac{2}{23}} & \frac{5}{\sqrt{46}} \\ \frac{-5}{\sqrt{26}} & 0 & 0 & \frac{1}{\sqrt{26}} \\ \frac{-2}{\sqrt{273}} & 0 & \sqrt{\frac{13}{21}} & \frac{-10}{\sqrt{273}} \\ \frac{-1}{\sqrt{483}} & \sqrt{\frac{21}{23}} & \frac{-4}{\sqrt{483}} & \frac{-5}{\sqrt{483}} \end{bmatrix}}_{\substack{V^T \\ 4\times 4}}$$

Last time: $C \in \mathbb{R}^{m\times n}$.

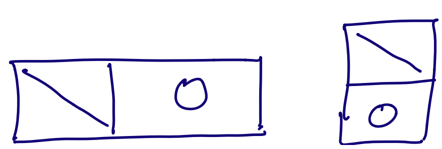


V such that $CV = U\Sigma$ — (*)

U : $m\times m$ orthonormal

V : $n\times n$ — " —

Σ : $m\times n$ "almost diagonal"



$$\boxed{C = U\Sigma V^{-1} = U\Sigma V^{T}}$$

Consider: $(C^TC) = S$

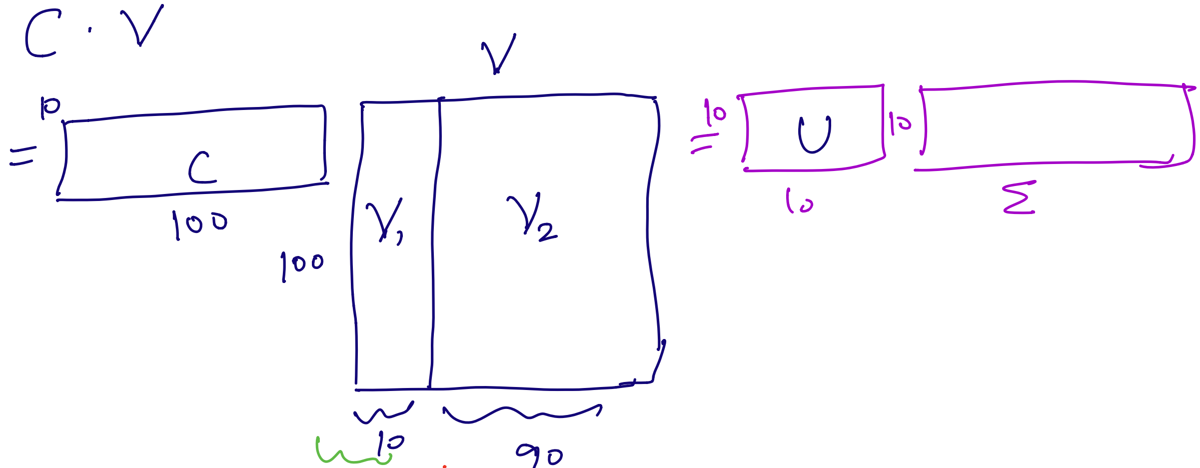
$C \in \mathbb{R}^{10\times 100}$
Rank $(C) = 10$

Choose V normalized e-vectors of C^TC.
Arranged $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_{100}$ so that

$\underbrace{\vec{v}_{11}, \vec{v}_{12} \dots \vec{v}_{100}}$ correspond to the 0 e-values.

↳ form a basis for the nullspace of both $N(C^TC)$ and $N(C)$.

$C \cdot V$



CV_1

Basis for the nullspace

$\{C\vec{v}_1, C\vec{v}_2 \cdots C\vec{v}_{10}\}$

→ basis for the columnspace of matrix C.

Question: Can we make this into an orthonormal basis? for col(C).

① Norm of $\|C\vec{v}_i\|^2 = \langle C\vec{v}_i, C\vec{v}_i \rangle$

$$= \vec{v}_i^T C^T C \vec{v}_i$$

only true cause $\vec{v}_i$ is evector of S $\longrightarrow$

$$= \vec{v}_i^T S \cdot \vec{v}_i$$

NOT e-vector of C

$$= \vec{v}_i^T \lambda_i \vec{v}_i$$

$$= \lambda_i \|\vec{v}_i\|^2 = \lambda_i$$

$\{C\vec{v}_1, C\vec{v}_2 \cdots C\vec{v}_{10}\}$

$\|C v_i\| = \sqrt{\lambda_i}$

$$\frac{\vec{a}}{\|\vec{a}\|}$$

→ Consider:

$$\underbrace{\left\{ \frac{C\vec{v}_1}{\sqrt{\lambda_1}}, \frac{C\vec{v}_2}{\sqrt{\lambda_2}}, \cdots, \frac{C\vec{v}_{10}}{\sqrt{\lambda_{10}}} \right\}}_{\text{now all are unit vectors ☺}} \qquad \lambda_i \neq 0.$$

Inner prod: $\left\langle \frac{C\vec{v}_i}{\sqrt{\lambda_i}}, \frac{C\vec{v}_j}{\sqrt{\lambda_j}} \right\rangle$

$$= \frac{\vec{v}_i^T C^T C \cdot \vec{v}_j}{\sqrt{\lambda_i \lambda_j}} = \frac{\vec{v}_i^T \lambda_j \vec{v}_j}{\sqrt{\lambda_i \lambda_j}}$$

$$= \sqrt{\frac{\lambda_j}{\lambda_i}} \underbrace{\langle \vec{v_i}, \vec{v_j} \rangle}_{0!} = 0 \,!$$

$\vec{v_i}, \vec{v_j}$ → e-vectors of S. orthonormal. } By Spectral thm.

__Now:__ $\left\{ \frac{C\vec{v_1}}{\sqrt{\lambda_1}}, \dots, \frac{C\vec{v_{10}}}{\sqrt{\lambda_{10}}} \right\}$

are all orthonormal!!

$$\boxed{\text{Define: } \quad \vec{u_i} = \frac{C\vec{v_i}}{\sqrt{\lambda_i}}} \quad \sqrt{\lambda_i} \to \sigma_i$$

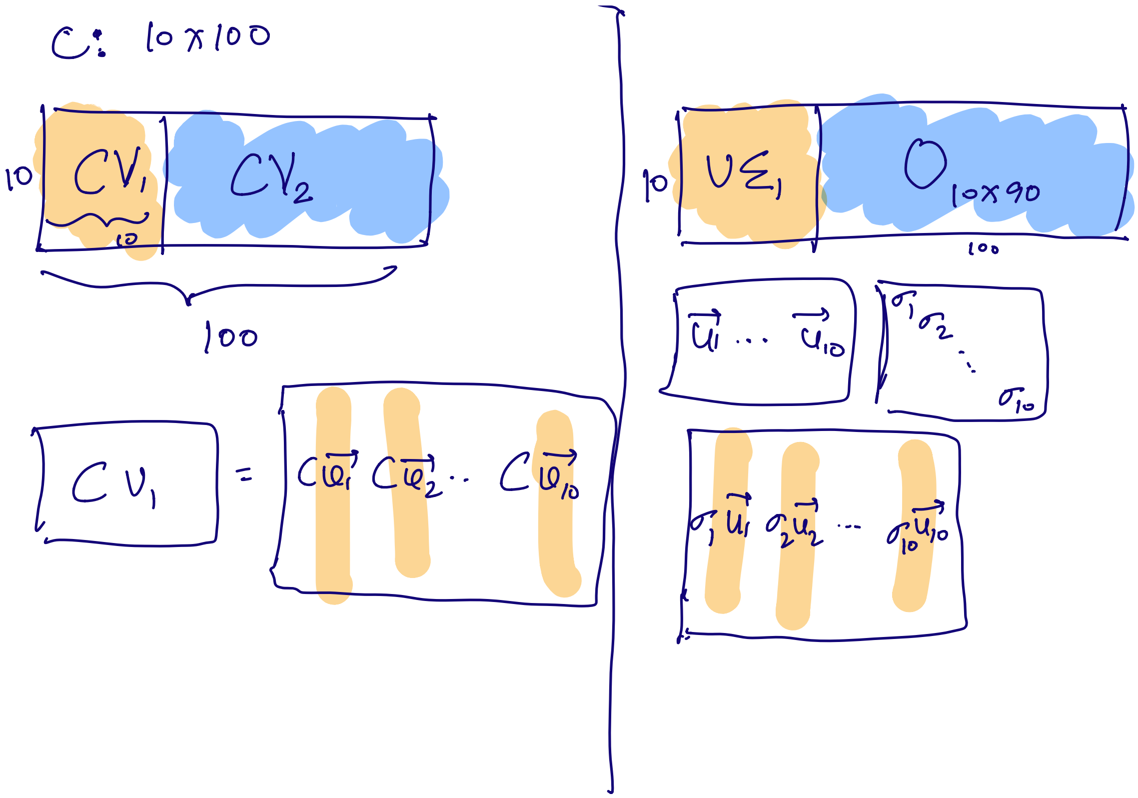
$\{ \vec{u_1}, \vec{u_2} \dots \vec{u_{10}} \}$ form an orthonormal basis for $Col(C)$ ☺

$$U = \begin{bmatrix} \vec{u_1}, & \vec{u_2} & \dots & \vec{u_{10}} \end{bmatrix}_{10 \times 10}$$

Want

$$C \underbrace{\begin{bmatrix} V_1 & V_2 \end{bmatrix}}_{100 \times (10 + 90)} = \underbrace{U}_{10 \times 10} \underbrace{\begin{bmatrix} \Sigma_1 & 0_{10 \times 90} \end{bmatrix}}_{10 \times (10 + 100)}$$

C: 10×100



$$C\vec{v}_i = \sigma_i \vec{u}_i$$

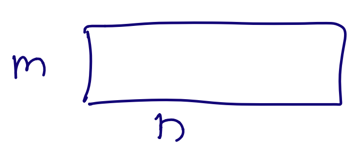
Choose $\sigma_i = \sqrt{\lambda_i}$!

$\Rightarrow$ SVD procedure will give us

$$CV = U\Sigma$$

or $$C = U\Sigma V^T.$$

Procedure for SVD: C. Full rank.



Rank(m)

(1) Compute $S = C^T C$.

Compute e-vectors S as V. (orthonormal)

$\rightarrow$ Use this to populate V matrix for SVD.

$\rightarrow$ $\vec{v}_1 \cdots \vec{v}_m$ correspond to non-zero evals.

$$\underbrace{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m}_{\text{non-zero}} > \underbrace{\lambda_{m+1} = \lambda_{m+2} = \cdots \lambda_n = 0}_{0}$$

$\rightarrow$ $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r > \underbrace{\lambda_{r+1} = \cdots = \lambda_n = 0}_{0}$

(2) $$\vec{u}_i = \frac{C \cdot \vec{v}_i}{\sqrt{\lambda_i}} \quad \text{for } \lambda_i \neq 0.$$

$$U = \begin{bmatrix} \vec{u}_1, & \vec{u}_2 & \cdots & \vec{u}_m \end{bmatrix}$$

$U_r = [\vec{u}_1, \vec{u}_2 \cdots \vec{u}_r]$

U basis for $\mathbb{R}^{m \times n}$

$\rightarrow$ Fill out Gram-Schmidt

$\Sigma_r : r \times r$

(3) $$\Sigma = \left[\begin{array}{cccccc|c} \sqrt{\lambda_1} & & & & & & \\ & \sqrt{\lambda_2} & & & & & \\ & & \ddots & & & & \\ & & & \sqrt{\lambda_r} & & & 0_{m \times (n-m)} \\ & & & & 0 & & \\ & & & & & \sqrt{\lambda_m} & \end{array} \right]$$

$\rightarrow$ $C = U \Sigma V^T$ SVD.

Note 13

What if: C was not full rank.

What if there were only 'r' non-zero e-values of C^TC. $r < m$, $r < n$.

"Compact form" SVD.

Reference: $A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 8 & 10 \end{bmatrix}_{2\times 4}$ Rank(A).

$$A = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}}_{\substack{U \\ 2\times 2}} \underbrace{\begin{bmatrix} \sqrt{230} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\substack{\Sigma \\ 2\times 4}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{46}} & \sqrt{\frac{2}{23}} & 2\sqrt{\frac{2}{23}} & \frac{5}{\sqrt{46}} \\ \frac{-5}{\sqrt{26}} & 0 & 0 & \frac{1}{\sqrt{26}} \\ \frac{-2}{\sqrt{273}} & 0 & \sqrt{\frac{13}{21}} & \frac{-10}{\sqrt{273}} \\ \frac{-1}{\sqrt{483}} & \sqrt{\frac{21}{23}} & \frac{-4}{\sqrt{483}} & \frac{-5}{\sqrt{483}} \end{bmatrix}}_{V^T : 4\times 4}$$

$$= \underbrace{\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}}_{\substack{U_r \\ 2\times 1}} \underbrace{\begin{bmatrix} \sqrt{230} \end{bmatrix}}_{\substack{\Sigma_r \\ 1\times 1}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{46}} & \sqrt{\frac{2}{23}} & 2\sqrt{\frac{2}{23}} & \frac{5}{\sqrt{46}} \end{bmatrix}}_{\substack{V_r^T \\ 1\times 4}}$$

$$= \underset{m\times r}{U_r} \quad \underset{r\times r}{\Sigma_r} \quad \underset{r\times n}{V_r^T}$$

Outer-Product Form:

$C \in \mathbb{R}^{m\times n}$

$\text{Rank}(C) = r$

$$\underset{m\times n}{C} = U \Sigma V^T$$

$$= \underset{m\times m}{\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}} \underset{m\times n}{\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0_{m\times(n-m)} \\ & & \sigma_r & \end{bmatrix}} \overset{n\times n}{\begin{bmatrix} - \vec{v}_1^T - \\ \vdots \\ - \vec{v}_n^T - \end{bmatrix}}$$

$$= \begin{bmatrix} \sigma_1 \vec{u}_1 & \sigma_2 \vec{u}_2 & \cdots & \sigma_r \vec{u}_r & \underbrace{\vec{0} \quad \vec{0} \cdots \vec{0}}_{\substack{n-m \\ \text{columns of } 0}} \end{bmatrix} V^T$$

$$= \underbrace{\begin{bmatrix} \sigma_1 \vec{u}_1 & \cdots & \sigma_r \vec{u}_r & \vec{0} \cdots \vec{0} \end{bmatrix}} \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix}$$

$$= \sum_{i=1}^{r} \sigma_i \underset{m\times 1}{\vec{u}_i} \underset{1\times n}{\vec{v}_i^T}$$

↳ r → rank of matrix

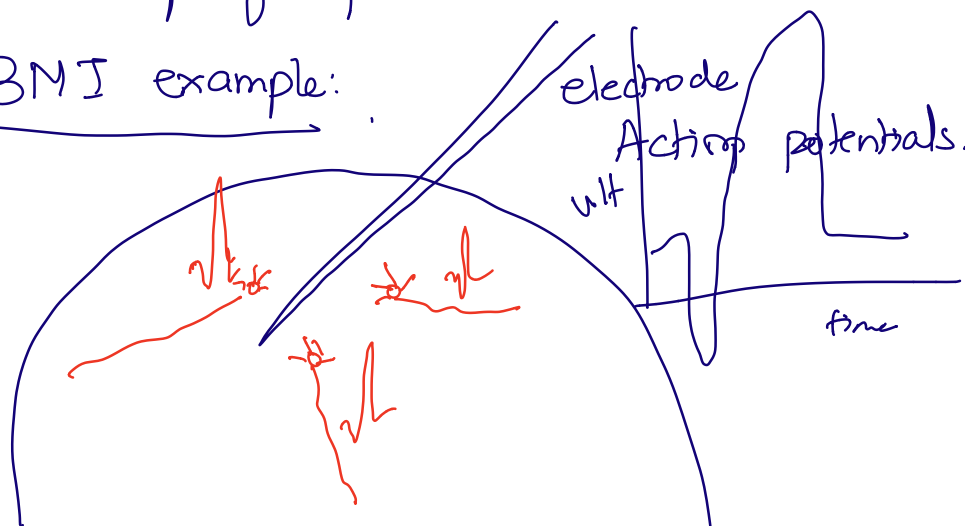
→ number non-zero singular values.

Why do we care???

$$A = U \left[\begin{array}{c|c} \begin{matrix} 100 & & & & & & & \\ & 90 & & & & & & \\ & & 50 & & & & & \\ & & & 1 & & & & \\ & & & & .5 & & & \\ & & & & & .2 & & \\ & & & & & & .01 & \\ & & & & & & & \ddots \end{matrix} & 0 \end{array}\right] V^T$$

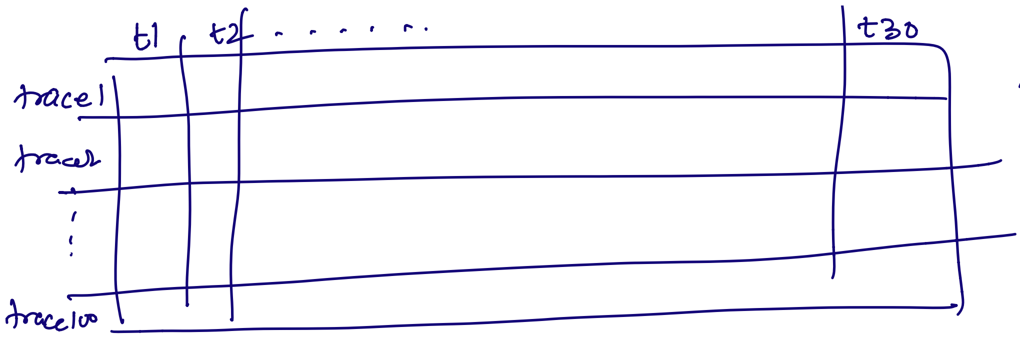
Underlying lower dim structure: SVD
can help you pull it out.

→ **BMI example:**

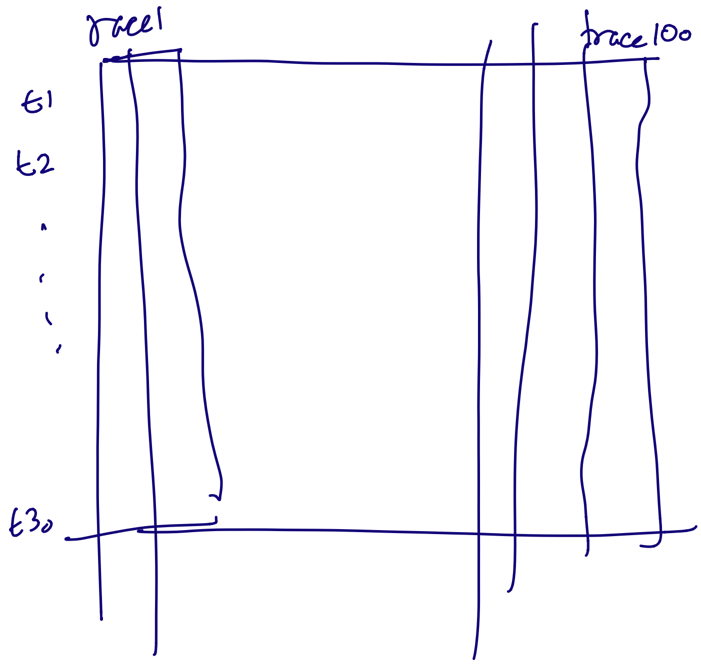


$$\vec{S} = \vec{S_A} + \vec{S_B} + \vec{S_C} + \vec{S_D} \dots$$

1. How many neurons are close to my electrode?
2. What do their individual action potentials look like??



traces as rows



traces as columns.