EECS 16B | Module 3, Lecture 5 | Apr 15, 2021

Today.

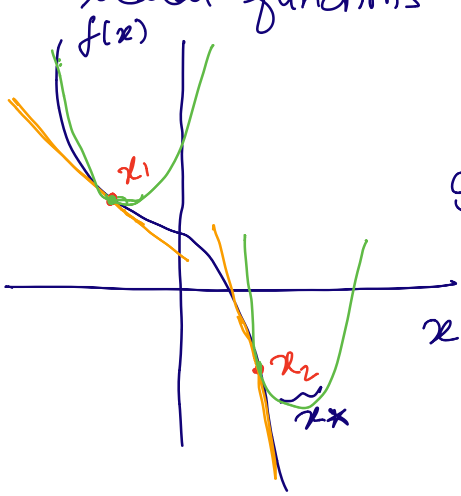
- Linearization : represent a non-linear funⁿ as a linear fun.
- Stationary / Equilibrium points / operating points.
- Pendulum.

Linearization.

$$\left.\begin{aligned} f(x) &= x^2 \\ f(x,u) &= x^2 u^3 \end{aligned}\right\}$$

$$\frac{d\vec{x}(t)}{dt} = f(\vec{x}, \vec{u}) = \underbrace{A\vec{x} + B\vec{u}}_{\text{linear formulation}}$$

Scalar functions (One variable)



$$\underbrace{f(x)}_{\text{general}} = f(\underbrace{x_*}_{\substack{\text{specific} \\ \text{"operating"} \\ \text{point}}}) + \underbrace{\text{stuff}}_{\substack{\text{depend on} \\ \text{how close} \\ x,\, x_* \text{ are}}}$$

How does f change in the neighbourhood of x_*?

Derivative of f at x_* $= \left.\dfrac{df}{dx}\right|_{x=x_*}$

Taylor Expansion

$$\boxed{f(x) = \underbrace{f(x_*)}_{\text{constant}} + \overbrace{\left.\frac{df}{dx}\right|_{x=x_*} (x - x_*)}^{\text{linear}}}$$

$$+ \left.\frac{d^2 f}{dx^2}\right|_{x=x_*} \frac{(x-x_*)^2}{2} \Big\} \text{ quadratic}$$

$$+ \left.\frac{d^3 f}{dx^3}\right|_{x=x_*} \frac{(x-x_*)^3}{6} \Big\} \text{ cubic}$$

$$+ \cdots\cdots$$

Scalar function case:

Linear approximation

$$f(x) \approx f(x_*) + f'(x_*)(x - x_*)$$

approx. equal

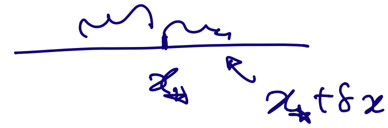
Multivariate function

$$\frac{dx}{dt} = f(x, u)$$

$$f(x,u) \approx f(x_*, u_*) + \underbrace{\text{stuff}}_{\text{linear in } x, \text{ and } u.}$$

<u>Example:</u> $f(x,u) = x^3 u^2$

How does f behave in a neighbourhood around (x_*, u_*)?



$$\underset{\text{variable}}{x} = \underset{\text{fixed}}{x_*} + \underbrace{\delta x}_{\text{delta } x}$$

delta x: a very small perturbation

$$u = u_* + \delta u$$

$$f(\overbrace{x_* + \delta x}^{x}, \overbrace{u_* + \delta u}^{u}) = (x_* + \delta x)^3 (u_* + \delta u)^2$$

$$= \left(x_*^3 + 3x_*^2 \cdot \delta x + 3x_*(\delta x)^2 + (\delta x)^3\right) \cdot \left(u_*^2 + 2u_*\delta u + (\delta u)^2\right)$$

$$= \underbrace{x_*^3 \cdot u_*^2}_{f(x_*, u_*)} + 3x_*^2 \cdot \delta x \cdot u_*^2 + x_*^3 (2u_*)\delta u + \text{stuff} \cdots$$

$$= f(x_*, u_*) + 3x_*^2 \cdot u_*^2 \cdot (x - x_*) + 2u_* x_*^3 (u - u_*) + \text{stuff.}$$

$3x_*^2 u_*^2$ $\qquad f(x,u) = \underline{x^3 u^2}$

$\hookrightarrow$ Pretend u is a constant, differential w.r.t. x

$$\frac{\partial f(x,u)}{\partial x} = u^2(3x^2)\Big|_{x=x_*,\, u=u_*} = 3x_*^2 u_*^2$$

$\hookrightarrow$ "Partial" derivative"

del f by del x

$$\frac{\partial f(x,u)}{\partial u} = x^3 2u\Big|_{x=x_*,\, u=u_*} = 2u_* x_*^3$$

<u>In general:</u>

$$f(x,u) \approx f(x_*,u_*) + \frac{\partial f}{\partial x}\Big|_{x=x_*,\, u=u_*}(x-x_*)$$

$$+ \frac{\partial f}{\partial u}\Big|_{x=x_*,\, u=u_*}(u-u_*)$$

Scalar-valued function of vector arguments

$$\underbrace{f(\vec{x}, \vec{u})}_{\in \mathbb{R}} \approx f(\vec{x}_*, \vec{u}_*)$$

$$+ \underbrace{A}_{\in \mathbb{R}^{1\times n}} \underbrace{(\vec{x} - \vec{x}_*)}_{\in \mathbb{R}^n}$$

$\vec{x} \in \mathbb{R}^n$

$\vec{u} \in \mathbb{R}^k$

$$+ \underbrace{B}_{\in \mathbb{R}^{1\times k}} \underbrace{(\vec{u} - \vec{u}_*)}_{\in \mathbb{R}^k}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow$$ How does f change with any component, assuming that all others are constant.

f changing with x_1 : $\dfrac{\partial f}{\partial x_1}$

f ——— x_2 : $\dfrac{\partial f}{\partial x_2}$

$\vdots$

$$A = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{1\times n}$$

Change w/ x_1 ... etc.

$$B = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial u_1} & \frac{\partial f}{\partial u_2} & \cdots & \frac{\partial f}{\partial u_k} \end{bmatrix}}_{\in \mathbb{R}^{1\times k}}$$

<u>Vector-valued function:</u> $\vec{f}(\vec{x},\vec{u}) \in \mathbb{R}^n$

$$\underbrace{\vec{f}(\vec{x},\vec{u})}_{\in \mathbb{R}^n} \simeq \vec{f}(\vec{x}_*,\vec{u}_*) + \underbrace{A}_{\in \mathbb{R}^{n\times n}}\underbrace{(\vec{x}-\vec{x}_*)}_{\in \mathbb{R}^n} + \underbrace{B}_{\in \mathbb{R}^{n\times k}}\underbrace{(\vec{u}-\vec{u}_*)}_{\in \mathbb{R}^k}$$

$$\vec{f}(\vec{x},\vec{u}) = \begin{bmatrix} f_1(\vec{x},\vec{u}) \\ f_2(\vec{x},\vec{u}) \\ \vdots \\ f_n(\vec{x},\vec{u}) \end{bmatrix}$$

$\to$ scalar valued function.
$\to$ " " "

"Jacobian"

$$A : \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & - & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

"Derivative" of $\vec{f}(\vec{x},\vec{u})$ w.r.t. $\vec{x}$

$$\frac{d\vec{f}}{d\vec{x}} = A$$

$$B: \underbrace{n\left\{\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_k} \\ \vdots & & \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_k} \end{bmatrix}\right.}_{k} \in \mathbb{R}^{n \times k}$$

How to choose $\vec{x}_*, \vec{u}_*$?

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}, \vec{u})$$

" Equilibrium / Steady state / Stationary points "

Find: $\frac{d}{dt}\vec{x}(t) = 0$ Stationary points.

$f(\vec{x}, \vec{u}) = 0$ find $\vec{x}, \vec{u}$

example:

$f(x, u) = 2x^2 - u$

$$\boxed{\frac{dx}{dt} = 2x^2 - u.}$$

$2x^2 - u = 0$

$\Rightarrow x = \pm\sqrt{u/2}$

$(\sqrt{u/2}, u) \longrightarrow$ Stationary pt 1.

$(-\sqrt{u/2}, u) \longrightarrow$ " —— " ——

Many. depend on u

Linearize around: $(\sqrt{4/2}, u)$.

Choose: $u = 2$. (an example)

Linearize around $(1, 2)$.

System:

$$\frac{dx}{dt} = 2x^2 - u = f(x, u).$$

$$f(x, u) = f(1, 2) + \left.\frac{\partial f}{\partial x}\right|_{x=1,\, u=2} (x-1) + \left.\frac{\partial f}{\partial u}\right|_{x=1,\, u=2} (u-2)$$

$$= 0 + \left.4x\right|_{x=1} (x-1) + \left.(-1)\right|_{u=2} (u-2)$$

$$= 4(x-1) - (u-2) = 4x - u - 2$$

$$\frac{\partial f}{\partial x} = \frac{\partial (2x^2 - u)}{\partial x} = 4x$$

$$\frac{\partial f}{\partial u} = \frac{\partial (2x^2 - u)}{\partial u} = -1$$

Linearization around $\left(\sqrt{\frac{u}{2}}, u\right) = (1, 2)$

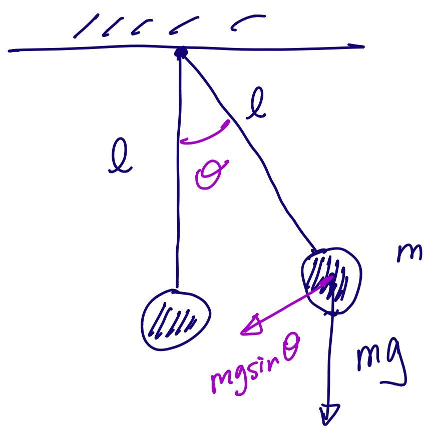
$$\underbrace{f(x, u) = 4x - u - 2}$$

Linear system!

UNSTABLE

$$\frac{dx}{dt} = Ax + Bu + c$$

Pendulum / Segway



$$ml\frac{d^2\theta}{dt} = -kl\frac{d\theta}{dt} - mg\sin\theta$$