EECS 16B — Module 3, Lecture 9 — Last lecture! — Apr 29, 2021.

Birds don't just fly, they fall down and get up.

Today: DFT continued.

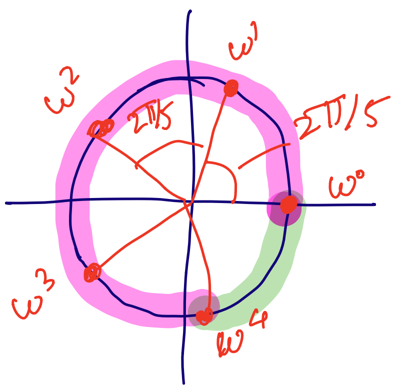
Last time:

Roots of unity: $z^N - 1 = 0 \qquad z \in \mathbb{C}$. $(z-1)(1+z+z^2+\dots+z^{N-1}) = 0$

N roots: $\underbrace{e^{j\frac{2\pi}{N}\cdot 0}}_{1}, e^{j\frac{2\pi}{N}\cdot 1} \dots \dots e^{j\frac{2\pi}{N}(N-1)}$.

$e^{j\frac{2\pi}{N}} = \omega$

Nth roots of unity: $\omega^0, \omega^1, \dots \omega^{N-1}$.



Property: (1) $1+\omega+\omega^2+\dots\omega^{N-1} = 0$, $\omega \neq 1$ (ω is Nth root of unity)

(2) $\omega^k = \omega^{-(N-k)}$

e.g. $\omega^4 = \omega^{-1}$: $e^{j\frac{2\pi}{5}(4)} = e^{j\frac{2\pi}{5}(-1)}$

How does this connect to frequency?

"How fast is you function changing?"

↳ How fast does your function go around the unit circle (complete 2π rotation)?

In one period, what is the angle covered by your function?

e.g. Consider: $N = 5$

$$f[k] = \omega^k = e^{j\frac{2\pi}{5}k}$$

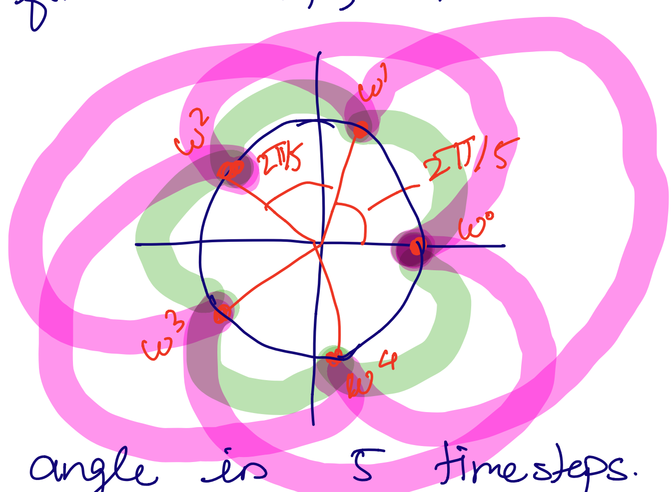
for $k = 0, 1, \cdots, 4$

$f[0] = 1$

$f[1] = \omega^1$

$\vdots$

$f[4] = \omega^4$



Complete 2π angle in 5 timesteps.

Angular frequency $\frac{2\pi}{5}$.

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e.g. $f[k] = \omega^{2k} = e^{j\frac{2\pi}{5}\cdot 2\cdot k}$

$f(0)$ ω^0, $f(1)$ ω^2, $f(2)$ ω^4, $f(3)$ $\omega^6 = \omega^5\cdot\omega^1 = \omega^1$, $f(4)$ $\omega^8 = \omega^5\cdot\omega^3 = \omega^3$

Step by angle $2\left(\frac{2\pi}{5}\right)$ every step

5 timesteps $\rightarrow$ go around unit circle twice!

4π angle in 5 timesteps.

Angular frequency: $\frac{4\pi}{5} = 2\left(\frac{2\pi}{5}\right)$.

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e.g. $f[k] = \omega^{0 \cdot k}$

$f(0),\ f(1),\ f(2),\ f(3),\ f(4)$

$\omega^0 = 1 \quad 1 \quad 1 \quad 1 \quad 1$

Angular frequency: $0\left(\frac{2\pi}{5}\right) = 0$.

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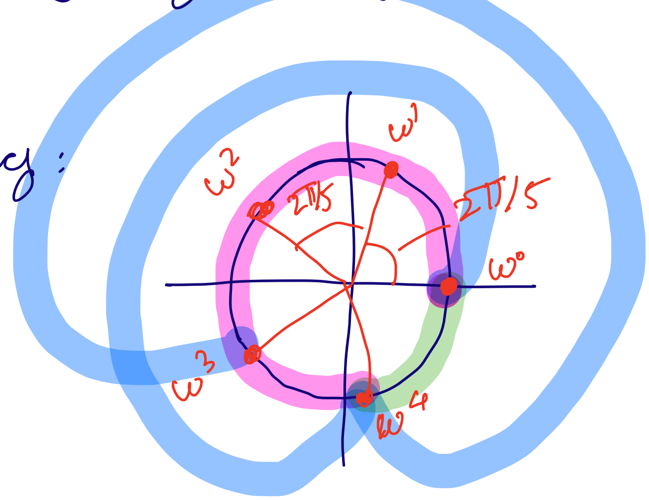
e.g. $f[k] = \omega^{4k}$

Jumping forward by angle $4\left(\frac{2\pi}{5}\right)$ each time step.

Angular frequency:

$4\left(\frac{2\pi}{5}\right)$

$\omega^4 = \omega^{-1}$



$f(0),\quad f(1),\quad f(2),\quad f(3),\quad f(4)$

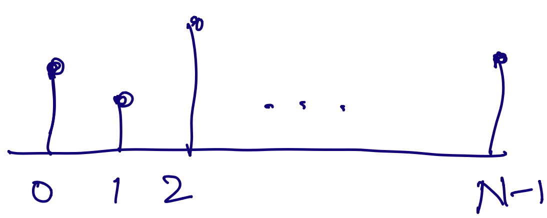
$\omega^0 \qquad \omega^4 \qquad \omega^8 = \omega^3 \qquad \omega^{12} = \omega^2 \qquad \omega^{16} = \omega^1$

$\omega^0 \qquad \omega^{-1} \qquad \omega^{-2} \qquad \omega^{-3} \qquad \omega^{-4}$

Angular frequency of: $\left(-\frac{2\pi}{5}\right)$

Goal: Represent time-domain signal in "frequency-domain" i.e. understand the constituent frequencies.

$x[k] \qquad k = 0, 1, \cdots, N-1.$



what are the frequencies?

e.g. $x[k] = e^{j\left(\frac{2\pi}{N}\right)k}$: Ang. freq: $\frac{2\pi}{N}$.

$x[k] = e^{j\left(\frac{2\pi}{N}\right)\cdot m \cdot k}$: Ang. freq: $\frac{2\pi m}{N}$

$$-\frac{2\pi(N-m)}{N}$$

$$0,\ \frac{2\pi}{N},\ 2\left(\frac{2\pi}{N}\right)\ \cdots\ ,\ (N-1)\frac{2\pi}{N}$$

Brings us to the DFT.

$\vec{x}$ of length N.

$\omega = e^{j2\pi/N}$

$$U = \frac{1}{\sqrt{N}}\begin{array}{c}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix} \\ \begin{array}{cccccc} 0 & \frac{2\pi}{N} & \frac{4\pi}{n} & \frac{3\cdot 2\pi}{N} & \cdots & \underbrace{(N-1)\frac{2\pi}{N}}_{-\frac{2\pi}{N}} \end{array}\end{array}$$

$$U = \begin{bmatrix} \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \end{bmatrix}$$

$$\vec{u}_k = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 \\ \omega^k \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix} \qquad k = 0, 1, \ldots N-1$$

U is orthonormal. $\rightarrow$ Columns are orthogonal to each other.

All are norm 1.

$\Rightarrow$ U forms a basis for $\mathbb{C}^N$

$\Rightarrow$ Every vector $\vec{x} \in \mathbb{C}^N$ can be written as a linear combination of the columns of U.

$$U \cdot \underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\text{capital } X} = \vec{x} = \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\text{time domain signal.}}$$

DFT coefficients.

frequency domain

$X[0]$ is considered to be the freq. domain coeff. corresponding to the angular freq. 0.

$X[1]$

$\frac{2\pi}{N}$

$$U \cdot \vec{X} = \vec{x}$$

$$\vec{X} = U^{-1} \vec{x}.$$

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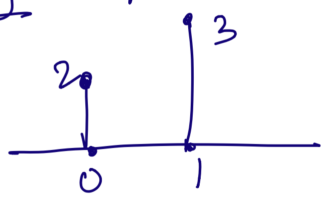
e.g.: $N = 2$ $x[0] = 2$, $x[1] = 3$.



time domain

$N = 2$.

$$w = e^{j\frac{2\pi}{2}} = e^{j\pi}$$

$$= \cos\pi + j\sin\pi = -1$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & w \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$U \cdot \vec{X} = \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \vec{X} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} 5/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{array}{l} \rightarrow \text{coeff. on freq. } 0 \\ \rightarrow \text{coef. on freq. } \pi \end{array}$$

# of frequencies that I decompose into is fixed by the signal length.

$$\cos(\omega t) = \underbrace{\tfrac{1}{2} e^{j2\pi t}}_{+1} + \underbrace{\tfrac{1}{2} e^{-j2\pi t}}_{\to -1}$$

$$\cos(\omega N) \quad \underbrace{\quad}_{1} \qquad \underbrace{\quad}_{N-1}$$

$$\sin(\omega t) = \frac{1}{2j} e^{j2\pi t} - \frac{1}{2j} e^{-j2\pi t}$$