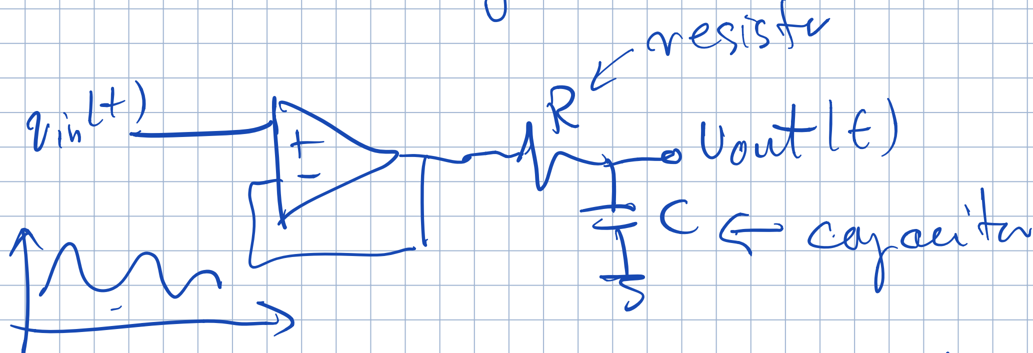
Lecture 5 EECS 16B

* RC filter review
* Solving systems of diff eqns.
(e.g. 2nd order diff. eqns with 2-capacitor RC filter)

Color organ lab



$$\frac{d}{dt} V_{out}(t) = -\frac{V_{out}(t)}{RC} + \frac{V_{in}(t)}{RC} \qquad RC = \tau$$

interested in $V_{out}(t)$ for $V_{in}(t) = V_{in} \cos(\omega t)$

guess: $V_{out}(t) = A \cdot \cos(\omega t + \theta)$

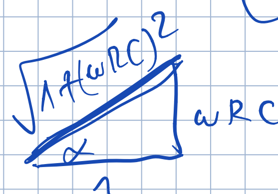
$$-A\omega \sin(\omega t + \theta) = -\frac{A}{RC} \cos(\omega t + \theta) + \frac{V_{in}}{RC} \cdot \cos(\omega t)$$

$$-A\omega RC \sin(\omega t + \theta) = -A \cos(\omega t + \theta) + V_{in} \cos(\omega t)$$

$$A\cos(\omega t+\theta) - A\omega RC\sin(\omega t+\theta) = V_{in}\cos(\omega t)$$

$$A\sqrt{1+(\omega RC)^2}\left(\underbrace{\frac{1}{\sqrt{1+(\omega RC)^2}}}_{\cos(\alpha)}\cos(\omega t+\theta) - \underbrace{\frac{\omega RC}{\sqrt{1+(\omega RC)^2}}}_{\sin(\alpha)}\sin(\omega t+\theta)\right)$$

$=$



$\left[\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\right]$

$$\tan(\alpha) = \frac{\dfrac{\omega RC}{\cancel{\sqrt{1+(\omega RC)^2}}}}{\dfrac{1}{\cancel{\sqrt{1+(\omega RC)^2}}}} = \omega RC$$

$$A\sqrt{1+(\omega RC)^2}\cdot\cos(\alpha+\omega t+\theta) = V_{in}\cos(\omega t)$$

$$\theta = -\alpha$$

$$\theta = -\tan^{-1}(\omega RC)$$

$$A\sqrt{1+(\omega RC)^2} = V_{in} \Rightarrow A = \frac{V_{in}}{\sqrt{1+(\omega RC)^2}}$$

$$\boxed{V_{out}(t) = \frac{V_{in}}{\sqrt{1+(\omega RC)^2}}\cdot\cos\left(\omega t - \tan^{-1}(\omega RC)\right)}$$

$$V_{out}(t) = \frac{V_{in}}{\sqrt{1+(\omega RC)^2}} \cdot \cos(\omega t - \tan^{-1}(\omega RC))$$

If $\omega \gg \frac{1}{RC}$ $\quad V_{out}(t) \xrightarrow{V_{in}} 0$

If $\omega \ll \frac{1}{RC}$ $\quad V_{out}(t) \approx V_{in} \cos(\omega t + \theta)$

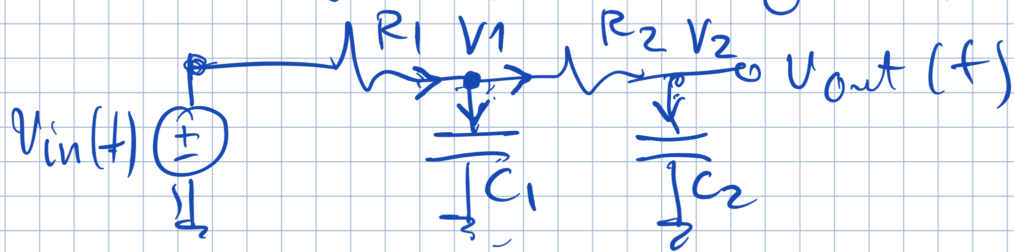
Attenuated signal has a decreased amplitude.

"RC filter" $\Rightarrow$ "low-pass"

$$\omega \gg \frac{1}{RC} \Rightarrow \omega RC \gg 1$$

$$\omega \ll \frac{1}{RC} \Rightarrow \omega RC \ll 1$$

* Solving systems of diff. eqns.



NVA:

$$\frac{V_{in}-V_1}{R_1} = C_1\frac{dV_1}{dt} + \frac{V_1-V_2}{R_2}$$

$$\frac{V_1-V_2}{R_2} = C_2\frac{dV_2}{dt}$$

$$\frac{dV_1}{dt} = -\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}\right)V_1 + \frac{V_2}{R_2C_1} + \frac{V_{in}}{R_1C_1}$$

$$\frac{dV_2}{dt} = \frac{1}{R_2C_2}V_1 - \frac{1}{R_2C_2}V_2$$

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} V_{in}$$

Know how to solve:

$$\frac{d}{dt}x = \lambda x + u$$

$R_1 = \frac{1}{3} M\Omega$, $R_2 = \frac{1}{2} M\Omega$, $C_1 = C_2 = 1\mu F$

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in}$$

How about "magic":

Assumption:

$V_{in} = 1V \quad t<0$

$\quad\quad 0V \quad t \geq 0$

$V_1(0) = 1V$

$V_2(0) = 1V$

Solving for $t \geq 0$

$$u_1 = V_2$$

$$u_2 = V_1 + 2V_2$$

$$\frac{d}{dt} u_1 = \frac{d}{dt} V_2 = 2V_1 - 2V_2 =$$

$$= 2u_2 - 4V_2 - 2V_2 =$$

$$= 2u_2 - 6V_2 = 2u_2 - 6u_1$$

$$\frac{d}{dt} u_2 = \frac{dV_1}{dt} + 2\frac{dV_2}{dt} =$$

$$= -5V_1 + 2V_2 + 4V_1 - 4V_2$$

$$= -V_1 - 2V_2 = -u_2$$

Wohoo!

$$\frac{d}{dt} u_2 = -u_2$$

know how to solve!

$$\frac{d}{dt} u_2 = -u_2 \Rightarrow u_2(t) = u_2(0) \cdot e^{-t}$$

$t \geq 0$

$$u_2(0) = V_1(0) + 2V_2(0) = 3\,V$$

$$\Rightarrow \boxed{u_2(t) = 3 \cdot e^{-t}}, \; t \geq 0$$

(assuming $V_{in}(t<0) = 1V$)

$$\frac{d}{dt} u_1(t) = 2u_2 - 6u_1$$

$$\frac{d}{dt} u_1(t) = -6u_1 + 6 \cdot e^{\overset{\text{“}s\text{”}}{-t}}, \; t \geq 0$$

Remember:

$$\frac{d}{dt} x(t) = \lambda x(t) + u(t)$$

where $u(t) = e^{st}$, $t \geq 0$

$$x(t) = K_2 e^{\lambda t} + \frac{e^{st}}{s - \lambda}$$

$$K_2 + \frac{1}{s-\lambda} = x(0)$$

$\lambda = -6$

$$u_1(t) = K_2 \cdot e^{-6t} + 6 \cdot \frac{e^{-t}}{-1+6}$$

$$u_1(t) = K_2 \cdot e^{-6t} + \frac{6}{5} e^{-t}$$

$$u_1(0) = K_2 + \frac{6}{5}$$

$$u_1(0) = V_2(0) = 1V \Rightarrow K_2 = -\frac{1}{5}$$

$$\boxed{u_1(t) = -\frac{1}{5} e^{-6t} + \frac{6}{5} e^{-t}}$$

have $u_1(t)$, $u_2(t)$ $\Rightarrow$ back solve for $V_1(t)$, $V_2(t)$

$$V_1(t) = u_2(t) - 2u_1(t)$$

$$= 3 \cdot e^{-t} - 2\left(-\frac{1}{5} e^{-6t} + \frac{6}{5} e^{-t}\right)$$

$$V_1(t) = 3 \cdot e^{-t} + \frac{2}{5} e^{-6t} - \frac{12}{5} e^{-t}$$

$$\boxed{V_1(t) = \frac{2}{5} e^{-6t} + \frac{3}{5} e^{-t}}$$

$$\boxed{V_2(t) = u_1(t) = -\frac{1}{5} e^{-6t} + \frac{6}{5} e^{-t}}$$

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{B} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}}_{B^{-1}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{B} \frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{B} \left(\underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in} \right)$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{B} \left(\underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}}_{B^{-1}} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in} \right)$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix}}_{BAB^{-1}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} V_{in}$$

upper-triangular

$\Rightarrow$ so can feed-back

we saw this in GE

Summary:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \qquad \vec{x}(0)$$

Native $\vec{x}$ coordinates:

"Nice" $\vec{\tilde{x}}$ coordinates:

$$\begin{tikzcd}[row sep=large, column sep=huge]
\text{variable} & \text{derivative} \\
\vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, bend right, "V^{-1}"'] \\
\vec{\tilde{x}} \arrow[u, bend right, "V"'] \arrow[r, "V^{-1}AV"'] & \frac{d}{dt}\vec{\tilde{x}} \arrow[u, bend right, "V"']
\end{tikzcd}$$

$$\vec{x}(t) = V\vec{\tilde{x}}(t)$$

$$\vec{\tilde{x}}(t) = V^{-1}\vec{x}(t)$$

$$\frac{d}{dt}\vec{\tilde{x}} = \frac{d}{dt}V^{-1}\vec{x}(t) = V^{-1}\frac{d}{dt}\vec{x}(t)$$

$$= V^{-1}Ax(t) = \underbrace{V^{-1}AV}_{}\,\vec{\tilde{x}}(t)$$

want this matrix to be "nice"

upper-triangular or even better - diagonal.