EECS 16B

Module 2, Lecture 2

Reminder:

MT1: March 15.

Today:

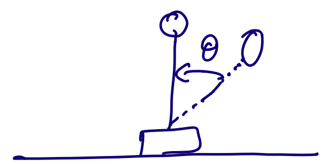
- Systems and states.
- Finish System ID
- Stability
- Feedback Control (Maybe)

"System" "State"

- Mars lander → x, y, z, position, velocity, acceleration

$$\left\{ \begin{aligned} S &= ut + \tfrac{1}{2} a t^2 \\ F &= m \cdot a \end{aligned} \right\}$$

- Inverted pendulum / cart-pole system.



- Disease spread $x(t)$

$$x(t+1) = \lambda \cdot x(t)$$

- Steam engine → coal, steam, temperature

Data-centric approach: $x(t+1) = \lambda x(t) + b \cdot u(t) + w(t)$

↳ Least squares → in scalar setting to "learn" λ, b as system parameters

→ Modern ML system

What about the vector case? A, B are matrices.

$$\vec{a}_1^{\,T} = [a_{11}\ a_{12} \cdots a_{1n}]$$

$$\underbrace{\begin{bmatrix} \vec{x}(t+1) \\ \\ \\ \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}}_{n\times n} \begin{bmatrix} \\ \vec{x}(t) \\ \\ \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \underbrace{\begin{bmatrix} \\ u(t) \\ \\ \end{bmatrix}}_{m\times 1}$$

A — n^2 unknown $B: n\times m$ — $n\times m$ unknowns

observe   observe   choose

Stage 1: Identify system:

Stage 2: Control

One approach: Stack all our unknowns into giant vector!

✓ Valid

Second approach: $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

$$\vec{x}(t+1) \quad \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ \vec{a}_n^{\,T} \end{bmatrix} \begin{bmatrix} \\ \vec{x}(t) \\ \\ \end{bmatrix} + \begin{bmatrix} \vec{b}_1^{\,T} \\ \vec{b}_2^{\,T} \\ \vdots \\ \vec{b}_n^{\,T} \end{bmatrix} \begin{bmatrix} \\ \vec{u}(t) \\ \\ \end{bmatrix}$$

$$x_1(t+1) = \vec{a}_1^T \cdot \vec{x}(t) + \vec{b}_1^T \cdot \vec{u}(t)$$

$$x_1(t+1) = \vec{x}^T(t)\,\vec{a}_1 + \vec{u}^T(t) \cdot \vec{b}_1$$

$$\begin{bmatrix} \vec{x}^T(t) & \vec{u}^T(t) \end{bmatrix} \underbrace{\begin{bmatrix} \vec{a}_1 \\ \vec{b}_1 \end{bmatrix}}_{n+m} = x_1(t+1)$$

one equation!

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{\vdots} \\ a_{1n} \\ b_{11} \\ b_{12} \\ \vdots \\ b_{1m} \end{bmatrix}$$

Need $n+m$ equations.

$n+m+1$ equations to do least squares + remove noise

$n+m+h$

$\boxed{n+m+h = l}$

$$\underbrace{\begin{bmatrix} \vec{x}(1)^T & \vec{u}(1)^T \\ \vec{x}(2)^T & \vec{u}(2)^T \\ \vdots & \vdots \\ \vec{x}(l)^T & \vec{u}(l)^T \end{bmatrix}}_{l \text{ observations}} \begin{bmatrix} \vec{a}_r \\ \vec{b}_r \end{bmatrix} = \begin{bmatrix} x_r(2) \\ x_r(3) \\ \vdots \\ x_r(l+1) \end{bmatrix}$$

rth row

l observations

$l > n+m+1$

$$\boxed{\begin{bmatrix} \vec{x}(1)^T & \vec{u}(1)^T \\ & \\ \vec{x}(l)^T & \vec{u}(l)^T \end{bmatrix} = D}$$

$$\left.\begin{aligned}
D \cdot \begin{bmatrix} \vec{a}_1 \\ \vec{b}_1 \end{bmatrix} &= \begin{bmatrix} x_1(2) \\ \vdots \\ x_1(l+1) \end{bmatrix} \\
D \cdot \begin{bmatrix} \vec{a}_2 \\ \vec{b}_2 \end{bmatrix} &= \begin{bmatrix} x_2(2) \\ \vdots \\ x_2(l+1) \end{bmatrix} \\
&\ \vdots \\
D \cdot \begin{bmatrix} \vec{a}_n \\ \vec{b}_n \end{bmatrix} &= \begin{bmatrix} x_n(2) \\ \vdots \\ x_n(l+1) \end{bmatrix}
\end{aligned}\right\} \text{One eq}^n \text{ for every row.}$$

Stack all your unknown columns next to each other!

$$D \quad \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} = \begin{bmatrix} x_1(2) & x_2(2) & \cdots & x_n(2) \\ x_1(3) & \vdots & & \vdots \\ \vdots & & & \vdots \\ x_1(l+1) & x_2(l+1) & & x_n(l+1) \end{bmatrix}$$

matrices

$$D \cdot P \approx S$$

$$P = (D^T D)^{-1} D^T S$$

Stage 1: "Learn" the system.

identify mass, length of racquet etc.

Stage 2: Control

"Stability"

known

DT: $x(t+1) = \lambda \cdot x(t) + u(t) + w(t).$

$u(t)$: input engineers choice

$w(t)$: nature's choice.

e.g: $x(t+1) = -2 \cdot x(t)$.

$x(0) = x(0)$.
$x(1) = -2\,x(0)$
$x(2) = (-2)^2\,x(0)$.
$\vdots$
$x(t) = (-2)^t \cdot x(0)$

What if $x(0)=0$?

e.g. $x(t+1) = \left(\frac{1}{2}\right) \cdot x(t)$
↳ "Stable"

"State-stability" Definition.

A system is stable if and only if there exists a K such that all $|x(t)| < K$ for all time t.

e.g. $x(t+1) = -2\,x(t) + u(t)$
$x(0) = 1$.

→ Also call unstable. to be stable for all $x(t)$, you need to be able to handle any $u(t)$

e.g: $x(t+1) = \frac{1}{2} \cdot x(t) + u(t)$.

New definition:

Bounded input bounded output stability

BIBO stability

BIBO stable if and only if when $\varepsilon \neq 0$

$|u(t)| \le \varepsilon$ for all t, then there exists K.

epsilon

$$\underbrace{|x(t)| \le K}_{\text{Bounded}} \quad \text{for all time } t.$$

"Bound"

Bounded

constant K.

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e.g.: $x(t+1) = \lambda \cdot x(t) + u(t)$. || $|u(t)| \le \varepsilon$

$|\lambda| > 1$, e.g $\lambda = (-2), 2 \dots$

Guess: Not BIBO stable.

Proof: Find some $u(t)$ such that the system blows up.

Try: $u(0) = \varepsilon$

$$u(t) = 0 \quad \text{for all } t > 0$$

$$x(0) = x(0)$$

$$x(1) = \lambda x(0) + u(0) = \lambda \cdot x(0) + \varepsilon$$

$$x(2) = \lambda(\lambda x(0) + \varepsilon) + 0$$

$$x(3) = \lambda^2(\lambda \cdot x(0) + \varepsilon)$$

$$\vdots$$

$$x(t) = \lambda^{t-1}(\lambda \cdot x(0) + \varepsilon).$$

$$|x(t)| = \underbrace{|\lambda^{t-1}|}_{\to \infty} \, |\lambda x(0) + \varepsilon|$$

unbounded

$\to$ BIBO unstable!

Example: $x(t+1) = \lambda \cdot x(t) + u(t)$. $\lambda = 1/2$

$|\lambda| < 1$, $|u(t)| \le \varepsilon$

Guess: BIBO stable.

Proof: Want to be bounded for all u's.

$$x[1] = \lambda \cdot x[0] + u[0]$$

$$\begin{aligned} x[2] &= \lambda \cdot x[1] + u[1] \\ &= \lambda\,[\lambda \cdot x[0] + u[0]] + u[1] \\ &= \lambda^2 \cdot x[0] + (\lambda \cdot u[0] + u[1]) \end{aligned}$$

Discussion

7A

(3A)

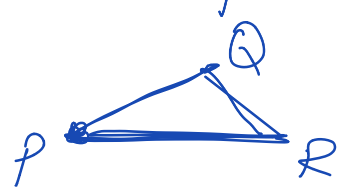
$$x[3] = \lambda^3 \cdot x[0] + (\lambda^2 u[0] + \lambda \cdot u[1] + u[2])$$

$$x[t] = \lambda^t \cdot x[0] + \sum_{i=0}^{t-1} \lambda^i u[t-1-i]$$

Want: $|x(t)| < K$

Take abs value on both sides.

$$|x(t)| = \left| \lambda^t \cdot x[0] + \sum_{i=0}^{t-1} \lambda^i u[t-1-i] \right|$$



$$|A + B| \le |A| + |B|$$

"Triangle inequality"

$$|x(t)| \leq |\lambda^t x[0]| + \left|\sum_{i=0}^{t-1} \lambda^i u[t-1-i]\right|$$

$\left|\frac{1}{2} x(0)\right| < |x(0)|$

$(-\frac{1}{2}) x(0) < |x(0)|$

$|\lambda| < 1$

$$\leq \underbrace{|x[0]|}_{\text{because } |\lambda| < 1} + \underbrace{\sum_{i=0}^{t-1} |\lambda^i| \, |u[t-1-i]|}_{\text{repeated application of triangle inequality}}$$

$$\leq \underbrace{|x[0]|} + \sum_{i=0}^{t-1} |\lambda^i| \cdot \varepsilon$$

$$= |x[0]| + \varepsilon \cdot \underbrace{\sum_{i=0}^{t-1} |\lambda^i|}$$

Case $\lambda = \frac{1}{2}$. $1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$

Geometric sum!

In general: $1 + |\lambda| + |\lambda^2| + \cdots = \frac{1}{1 - |\lambda|}$

if $|\lambda| < 1$

$$\leq |x[0]| + \varepsilon \left(\frac{1}{1 - |\lambda|}\right) = K$$

Bound!

BIBO stable!!

Office hours: