EECS 16B — Module 2, Lecture 3

Today:
- Continue Stability (BIBO)
  - Discrete Time
  - Continuous Time
- Feedback Control / Stabilization / Eigenvalue placement
- Controllability.

Discrete Time System

$$x[t+1] = \lambda x[t] + \underbrace{u[t]}_{\text{engineers choice.}} + \underbrace{w[t]}_{\text{nature}}$$

BIBO stable: System is BIBO stable if & only if when $|u[t]| < \varepsilon$ and $|w[t]| < \varepsilon$ for all t then there exists a K such that all $|x[t]| < K$ for all t.

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for all : $\forall$

exists : $\exists$

Cases: $|\lambda| > 1$: unstable

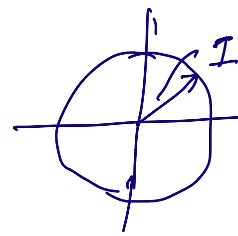
$|\lambda| < 1$: stable

$|\lambda| = 1$: ??.

$$|u(t) \le \varepsilon|$$

Case: $|\lambda| = 1$

$$\Rightarrow \lambda = e^{j\theta}$$



$\lambda \in$ unit. circle

Complex plane.

$$x(t) = \lambda^t \cdot x[0] + \sum_{k=0}^{t-1} \lambda^k u[t-1-k]$$

$$= e^{j\theta t} x[0] + \sum_{k=0}^{t-1} u[t-1-k] \cdot \left(e^{j\theta}\right)^k$$

If $\lambda = 1$

$$x(t) = e^{j\theta t} x[0] + \underbrace{\sum_{k=0}^{t-1} u[t-1-k]}$$

$$= \underbrace{e^{j\theta t} \cdot x[0] + t \cdot \varepsilon}_{\text{Blows up.}}$$

Choose $u(t) = \varepsilon$ $\forall t$

Unstable!

Choose:

$$x(t) = e^{j\theta t} x[0] + \sum_{k=0}^{t-1} \varepsilon \cdot \overbrace{e^{-j\theta k}}^{u[t-1-k]} \cdot e^{j\theta k}$$

$$u(t-i-k) = e^{-j\theta k}$$

$$m = t-i-k \quad \Rightarrow \quad k = t-i-m$$

$$\Rightarrow \quad u(m) = \varepsilon \cdot e^{-j\theta k} = \varepsilon e^{-j\theta(t-i-m)}.$$

$$x(t+1) = e^{j\theta t} \cdot x[0] + \sum_{k=0}^{t-1} \varepsilon$$

$$x(t+1) = e^{j\theta t} x[0] + \varepsilon \cdot t.$$

$\hookrightarrow$ Unstable!!

Discrete-Time Systems:

$\lvert\lambda\rvert > 1$	unstable
$\lvert\lambda\rvert < 1$	stable
$\lvert\lambda\rvert = 1$	marginally stable / unstable.

Vector system:

$$\vec{x}[t+1] = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \vec{x}[t] + \vec{u}[t] + \vec{w}[t]$$

$\hookrightarrow$ BIBO unstable

BIBO stability: All $|\text{eigenvalues}| < 1$.

$$\vec{x}[t+1] = A\vec{x}[t] + \vec{u}[t] + \vec{w}[t].$$

$$A = V \Lambda V^{-1}$$

Λ = diagonal matrix.

Eigenvalues of A are eigenvalues of Λ and they are along the diagonal!

$$\vec{x}[t+1] = V \Lambda V^{-1} \cdot \vec{x}[t] + \vec{u}[t] + \vec{w}[t]$$

$$V^{-1}\vec{x}[t+1] = \Lambda V^{-1}\vec{x}[t] + V^{-1}\vec{u}[t] + V^{-1}w[t]$$

$V^{-1}\vec{x}[t+1] = \vec{\tilde{x}}[t+1]$ and so on.

$$\vec{\tilde{x}}[t+1] = \underbrace{\Lambda}_{\text{diagonal!}} \vec{\tilde{x}}[t] + \vec{\tilde{u}}[t] + \vec{\tilde{w}}[t]$$

→ Find e-vals of A – if all have $|\lambda| < 1$ then → BIBO stable!

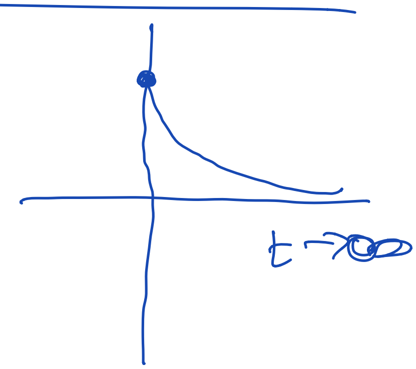
Continuous-Time systems

$$\frac{d}{dt} x(t) = \lambda \cdot x(t) + u(t).$$

$$x(t) = e^{\lambda t} \cdot x(0) + \int_0^t e^{\lambda(t-\tau)} u(\tau) \cdot d\tau.$$

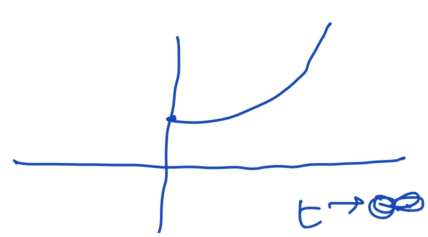
$$\frac{d}{dt} x(t) = -2 \cdot x(t)$$

$$x(t) = e^{-2t} \cdot x(0)$$



$$\frac{d}{dt} x(t) = 2 \cdot x(t)$$

$$x(t) = e^{2t} x(0)$$



Continuous time:

$Re\{\lambda\} > 0$ unstable.

$Re\{\lambda\} < 0$ stable.

$Re\{\lambda\} = 0$ marginally ~~unstable~~/stable

New control system.

1. Identify the system: learn parameters
   ↳ sys ID. → model.

2. Once we have the model, is the system stable or not?

3. What can we do if your system is not stable? Can we make it stable?
   ↳ Feedback Control.

4. Can we drive the system to where we want to go?
   ↳ Controllability.

Feedback: Op Amps

Discrete-Time:

Open loop dynamics $\Big\{ \vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t] + \vec{w}[t]$

A unstable

|eigenvalues| > 1

Feedback: Choose $\vec{u}(t)$ as a function of $\vec{x}(t)$ so that you can adjust the system behavior.

$$\underbrace{\text{Choose } \vec{u}(t) = F \cdot \vec{x}(t)}$$

Substituting:

$$\vec{x}[t+1] = A\vec{x}[t] + B \cdot F \cdot \vec{x}[t] + \vec{w}[t]$$

Closed loop dynamics $\Big\{ \vec{x}[t+1] = (A + BF)\vec{x}(t) + \vec{w}(t)$

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e.g. $\underline{x[t+1] = 2x[t] + u(t) + w[t]}$

$u(t) = f \cdot x(t)$.

Want eigenvalue λ_0

Choose: $f = \lambda_0 - 2$.

$$x[t+1] = 2x[t] + (\lambda_0 - 2)\cdot x[t] + w[t]$$

$$x[t+1] = \lambda_0 \cdot x[t] + w[t]$$

"state feedback"

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e.g: Vector systems.

Closed dynamics:

$$\vec{x}[t+1] = \underbrace{(A+BF)}\cdot\vec{x}(t) + \vec{w}(t).$$

Say we wanted $A+BF = G$

$\Rightarrow \quad BF = G - A$

$\Rightarrow \quad F = (B^{-1})(G-A)$ if B is invertible!

What if B is not invertible?

$B = \vec{b}$ vector.

eg: $$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t] + \vec{w}[t]$$

$\vec{x}[t] \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times 2}$, $\vec{b} \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$

$\vec{w} \in \mathbb{R}^{2\times 1}$

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \longrightarrow \underline{\text{eigenvalues?}}$$

$\lambda_1 = -1, \ \lambda_2 = 3$

$$u(t) = F \cdot \vec{x}[t]$$

$$u(t) = [f_1 \ \ f_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Want to find f_1, f_2 so that the system closed-loop is stable.

$$A + BF = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1 \ \ f_2]}_{\begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 1 \\ 3+f_1 & 2+f_2 \end{bmatrix}$$

e-vals?

$$(A + BF - \lambda I): \begin{bmatrix} -\lambda & 1 \\ 3+f_1 & 2+f_2-\lambda \end{bmatrix}$$

Determinant:

$$(-\lambda)(2+f_2-\lambda) - 1(3+f_1) = 0$$

$$\lambda^2 - (2+f_2)\cdot\lambda - (3+f_1) = 0$$

<u>Want:</u> λ_1, λ_2 roots

$$\lambda_1 \cdot \lambda_2 = -(3+f_1)$$

$$\lambda_1 + \lambda_2 = (2+f_2)$$

$$(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2) + \lambda_1\lambda_2$$

$\Rightarrow$ We can achieve ANY λ_1, λ_2 by choice of f_1, f_2 !!!

"Eigenvalue placement"

<u>e.g.</u> $\vec{x}(t+1) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \vec{w}(t)$

$F = [f_1 \ \ f_2]$

$$A + B \cdot F = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ f_1 & 3+f_2 \end{bmatrix}$$

e-vals:

$$\begin{bmatrix} 2-\lambda & 0 \\ f_1 & 3+f_2-\lambda \end{bmatrix}$$

determinant:

$$(2-\lambda)(3+f_2-\lambda) - 0 \cdot f_1 = 0$$

$$\Rightarrow \lambda^2 - \lambda(2+3+f_2) + 2(3+f_2) - 0 = 0$$

$$\Rightarrow \lambda^2 - \lambda(5+f_2) + 2(3+f_2) = 0$$

want λ_1, λ_2:

$$\lambda_1 \lambda_2 = 2(3+f_2)$$

$$\lambda_1 + \lambda_2 = 5+f_2$$