EECS 16B   Module 2, Lecture 4

March 9, 2021

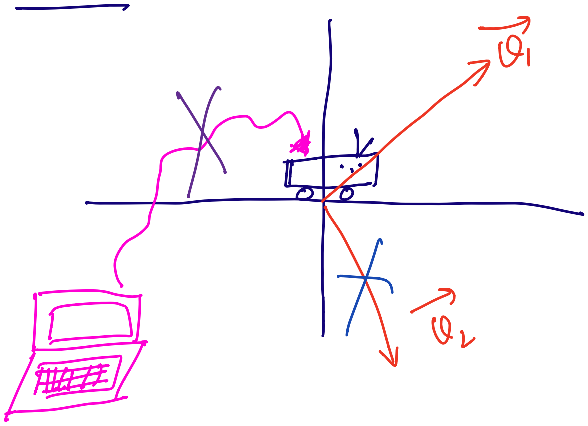
- March 8 – International Womens Day.
- March 15 – Midterm
- Contest (TI)

- Last time
- Stability
- Feedback Control (i.e. Magic).
  ↳ Use linear functions of the state to "place eigenvalues" and stabilize.

Today: Controllability

↳ Similar to stability but different.

Consider



Can car get anywhere in $\mathbb{R}^2$?
↳ Yes, if LI.

Can you still get anywhere in $\mathbb{R}^2$ with just $\vec{v}_1$?

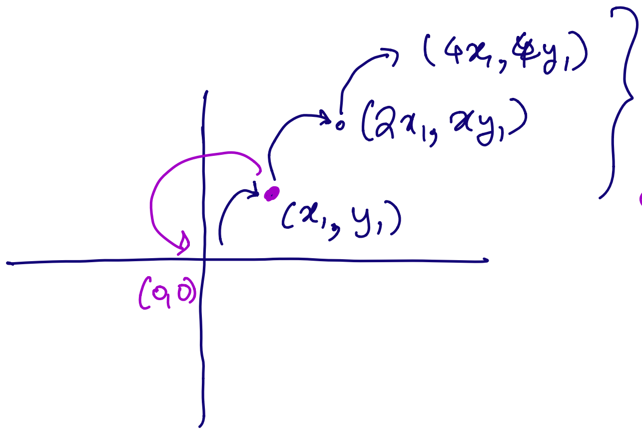
→ No!

Consider dynamics $\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$

① $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → Not Controllable

$\vec{x}[t+1] = A \cdot \vec{x}[t]$

(2) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



with no input.

→ Use input to take system $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$B \cdot [u(t)] = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} 0 \\ u \end{bmatrix}$$

$$\vec{x}[t+1] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \vec{u}(t)$$

Not controllable.

(3) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. X

(4) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

✓ Controllable ✓

$\vec{u}(t)$: 2×1 vector

(5) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

General system:

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t] + \cancel{\vec{w}[t]}.$$

set $\vec{w}$ to 0

$$\vec{x}[2] = A\left[A\cdot\vec{x}[0] + B\vec{u}[0]\right] + B\vec{u}[1]$$

$$= A^2\cdot\vec{x}[0] + AB\vec{u}[0] + B\vec{u}[1] \quad \Big\} \text{ span}\{B, AB\}$$

$$\vec{x}[t] = A^t\vec{x}[0] + A^{t-1}\cdot B\cdot\vec{u}[0] + A^{t-2}\cdot B\,\vec{u}[1] + \cdots + B\cdot\vec{u}[t-1]$$

$$\boxed{\vec{x}(t) = \underbrace{A^t\cdot\vec{x}[0]}_{\text{initial condition}} + \boxed{\sum_{i=0}^{t-1} A^{t-1-i}B\cdot\vec{u}[i]}}.$$

engineer can't touch this

$\vec{u}[1]\cdot\vec{v}_1 + \vec{u}[2]\vec{v}_2 \;\cdots$

$$\boxed{\text{Span}\{B, AB, A^2B, \ldots\}}$$

Example: Reaching somewhere in one step.

$$\vec{x}[1] = A\cdot\vec{x}[0] + B\vec{u}[0]$$

Want: $\vec{x}[1] = \vec{x}_*$

$$\vec{x}_* = A\vec{x}[0] + B\vec{u}[0].$$

$$\Rightarrow \underbrace{B}\cdot \vec{u}[0] = \vec{x}_* - A\vec{x}[0]$$

If B is invertible, I can solve for $\vec{u}[0]$

If $\vec{x}_* - A\vec{x}[0] \in \text{Range}(B) \rightarrow$
the I can find sol^{n}.

Definition: Controllability. $\vec{x} \in \mathbb{R}^n$

A system with state $\vec{x}(t)$ is controllable if at some time t $\vec{x}(t)$ can be brought to any point in $\mathbb{R}^n$.

To get to some $\vec{x}_*$

$$\vec{x}_* - A^t \cdot \vec{x}[0] = \underbrace{\sum_{i=0}^{t-1} A^{t-1-i} B \vec{u}[i]}_{\text{span}(B, AB, \cdots A^{t-1}B)}$$

want this $= \mathbb{R}^n$.

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e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$t = 1$ $\quad \text{span}\{B\} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \neq \mathbb{R}^2$

$t = 2$ $\quad$ $\text{span}\{B, AB\}$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \mathbb{R}^2 \checkmark$$

$\checkmark$ is Controllable $\checkmark$.

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e.g. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$t = 1$ $\quad$ $\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \rightarrow \neq \mathbb{R}^2$

$t = 2$ $\quad$ $\text{span}\left\{\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B}, \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{AB}\right\}$

$$= \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right\}$$

$$= \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \neq \mathbb{R}^2$$

$t = 3$ $\quad$ $\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \underbrace{A^2B}_{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}}\right\}$

$$= \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$\neq \mathbb{R}^2$

$t = 4$ You did this ☹

Conside $B = \vec{b}$ for convenience.

Thm: $\text{span}\{\vec{b}, A\vec{b}, \ldots A^{n-1}\vec{b}\}$

If $A^n\vec{b}$ is linearly dependent on $\{\vec{b}, A\vec{b}, \ldots A^{n-1}\vec{b}\}$, then $A^{n+1}\vec{b}$ is also linearly dependent on $\{\vec{b}, A\vec{b}, \ldots A^{n-1}\vec{b}\}$.

Proof:

Known:

$$A^n\vec{b} = \alpha_0\vec{b} + \alpha_1(A\vec{b}) + \alpha_2(A^2\vec{b}) + \cdots + \alpha_{n-1}(A^{n-1}\vec{b})$$

Want to show:

$$A^{n+1}\vec{b} = \beta_0\vec{b} + \beta_1(A\vec{b}) + \cdots + \beta_{n-1}(A^{n-1}\vec{b})$$

Such $\beta_1, \beta_2, \ldots \beta_{n-1}$ exist.

$$A^{n+1}\vec{b} = A\left(A^{n}\vec{b}\right)$$

$$= A\left(\alpha_0 \vec{b} + \alpha_1 (A\vec{b}) + \cdots \alpha_{n-1}\left(A^{n-1}\vec{b}\right)\right)$$

$$= \alpha_0 A\vec{b} + \alpha_1 A^2\vec{b} + \cdots + \alpha_{n-1} A^{n}\vec{b}.$$

$$= \alpha_0 A\vec{b} + \alpha_1 A^2\vec{b} + \cdots + \alpha_{n-2} A^{n-1}\vec{b}$$

$$+ \alpha_{n-1}\left(\alpha_0 \vec{b} + \alpha_1 (A\vec{b}) + \cdots + \alpha_{n-1}(A^{n-1}\vec{b})\right)$$

$$= \alpha_{n-1}\alpha_0 \vec{b} + (\alpha_0 + \alpha_{n-1}\alpha_1) A\vec{b} + \cdots$$

$$+ (\alpha_{n-2} + \alpha_{n-1}\cdot\alpha_{n-1}) A^{n-1}\vec{b}$$

$\blacksquare$ QED.

$\Rightarrow A^{n+1}\vec{b}$ is also a lin. comb. of

$$\{\vec{b}, A\vec{b}, \ldots, A^{n-1}\vec{b}\}.$$

To check controllability, once the span stops growing $\rightarrow$ we never grow again!

Could you grow indefinitely?

$\vec{x}(t) \in \mathbb{R}^n \rightarrow$ then the largest the span could be is $\mathbb{R}^n$.

$\rightarrow$ If $\text{span}\{\vec{b}, A\vec{b}, \ldots, A^{n-1}\vec{b}\} = \mathbb{R}^n$

$\Rightarrow$ system is controllable.

if not! system is not controllable!

$$\text{Rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$$

✓ Controllable.

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\text{Rank}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \rightarrow 2\ ?$$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$