EECS 16B

Module 2/3, Lecture 7

Today:

- Upper Triangulation (continued) [What if we can't diagonalize?].
  - E-vals of an UT matrix.
    - $\hookrightarrow$ $M = U^{-1} T U$. U orthonormal, T is upp. tri.
    - M, T have same e-vals
- Evals of T are along diagonal.

Introduce Induction.

- Back to BIBO stability ← only depends on e-vals.
- Spectral theorem for symmetric matrices. ←

3x3 case: $M \in \mathbb{R}^{3\times3}$.

(1) $\vec{v_1}$ be an eigenvector.

$$M\vec{v_1} = \lambda_1 \vec{v_1} \qquad \lambda_1 : \text{e-val of } M.$$

$\longrightarrow$ Choose $\|\vec{v_1}\| = 1$

(2) Try U: $\left[\vec{v_1}, \overset{\vec{r_1}}{\quad}, \overset{\vec{r_2}}{\quad}\right]$

U: orthonormal basis.

Construct $\vec{r_1}, \vec{r_2}$ using Gram Schmidt

s.t. $\langle \vec{v_1}, \vec{r_1} \rangle = \langle \vec{v_1}, \vec{r_2} \rangle = \langle \vec{r_1}, \vec{r_2} \rangle = 0$

$\|\vec{r_1}\| = \|\vec{r_2}\| = 1$.

$\vec{r_1} \in \mathbb{R}^3$

Define: $R = \left[\vec{r_1} \;\; \vec{r_2}\right]$ $\quad \boxed{R \in \mathbb{R}^{3\times2}}$

$$U = \begin{bmatrix} \vec{u}_1 & R \end{bmatrix} \qquad R^T = \begin{bmatrix} -\vec{r}_1^{\,T}- \\ -\vec{r}_2^{\,T}- \end{bmatrix}$$

Try: $U^{-1} M U$

$$= \begin{bmatrix} \vec{u}_1 & R \end{bmatrix}^{-1} M \begin{bmatrix} \vec{u}_1 & R \end{bmatrix}$$

$$= \begin{bmatrix} -\vec{u}_1^{\,T}- \\ -R^T- \end{bmatrix} M \begin{bmatrix} \vec{u}_1 & R \end{bmatrix}$$

$$= \begin{bmatrix} -\vec{u}_1^{\,T}- \\ R^T \end{bmatrix} \begin{bmatrix} M\vec{u}_1 & MR \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}_1^{\,T} M \vec{u}_1 & \vec{u}_1^{\,T} M R \\ \underbrace{R^T M \vec{u}_1} & \boxed{R^T M R} \end{bmatrix}$$

$$R^T M \vec{u}_1$$
$$= R^T \lambda_1 \vec{u}_1$$
$$= \lambda_1 R^T \vec{u}_1$$
$$= \lambda_1 \begin{bmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \end{bmatrix} \vec{u}_1$$

$$= \begin{bmatrix} \lambda_1 & \vec{u}_1^{\,T} M R \\ \boxed{\begin{matrix} 0 \\ 0 \end{matrix}} & \underbrace{\boxed{R^T M R}} \end{bmatrix}$$

$R^T: 2\times 3$ $\qquad$ $R: 3\times 2$.

$M: 3\times 3$

$R^T M R: 2\times 2$.

Need : $\boxed{R^TMR}$ also should be upper triangular!

Reduced problem. 2×2 Matrix R^TMR.

There exist U_2 such that

$$(U_2)^{-1} \underbrace{(R^TMR)}_{2\times 2} U_2 = \underset{\text{Upper tria.}}{T_2}$$

$$= \underbrace{U_2^T R^T}MR\, U_2$$

$$= \underbrace{(RU_2)^T}_{\text{matrix}} M \underbrace{(RU_2)}_{\text{matrix}} \rightarrow \text{upper triangular!}$$

Instead of choosing $\underbrace{U = [\vec{u_1} \quad R]}_{\text{first guess.}}$

Choose:

$$U = [\vec{u_1} \quad RU_2]$$

$U^{-1} M U$

$= [\vec{u_1} \quad RU_2]^{-1} M [\vec{u_1} \quad RU_2]$

Check:

Is $[\vec{u_1} \quad RU_2]$ orthonormal? $\rightarrow$ YES.

$\langle \vec{u_1}, \underbrace{RU_2} \rangle = \vec{u_1}^T R U_2$

$= 0$ ✓

Because $\vec{u_1}^T R = [0 \quad 0]$

Check:

$\langle RU_2, RU_2 \rangle = (RU_2)^T R U_2.$

$R = {}^{3\times 2}$

$= U_2^T R^T R U_2$

$R^T R$
$(2\times 3) \times (3\times 2)$

$= U_2^T I_{2\times 2} U_2$

$= U_2^T U_2$

$= I_{2\times 2}.$

Back to checking if modified

$$U = [\vec{u}_1 \;\; R U_2] \text{ actually works.}$$

$$U^{-1} M U = \begin{bmatrix} \vec{u}_1^{\,T} \\ U_2^T R^T \end{bmatrix} M \, [\vec{u}_1 \;\; R U_2]$$

$$= \begin{bmatrix} \vec{u}_1^{\,T} M \vec{u}_1 & \vec{u}_1^{\,T} M R U_2 \\ U_2^T R^T M \vec{u}_1 & \boxed{U_2^T R^T M R U_2} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \\ 0 & \underbrace{\boxed{\begin{matrix} & \\ 0 & \end{matrix}}}_{2\times 2} \end{bmatrix}$$

$$U_2^T R^T M \vec{u}_1 = U_2^T R^T \lambda_1 \vec{u}_1 = 0$$

$\Rightarrow$ Hence: $U = [\vec{u}_1 \;\; R U_2]$

is a basis that upper triangularizes M.

Started 3×3 $\longrightarrow$ reduced it to the 2×2 case and used sol^n for 2×2 $\longrightarrow$ construct back 3×3 solution.

General case: $M \in \mathbb{R}^{n\times n}$.

$$M\vec{u_1} = \lambda_1 \vec{u_1}$$

Pick $U = [\vec{u_1} \quad R]$ R constructed by Gram Schmidt

$$U^T M U = \begin{bmatrix} \vec{u_1}^T \\ R^T \end{bmatrix} M \begin{bmatrix} \vec{u_1} & R \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & \vec{u_1}^T M R \\ \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} & \underbrace{\boxed{R^T M R}}_{(n-1)\times(n-1)} \end{bmatrix}$$

Reduce to triangularizing $(n-1) \times (n-1)$ matrix !

We know:

2×2 matrix UT ✓

3×3 matrix UT ✓

If $(n-1) \times (n-1)$ matrix can be upper triangularized, then an $(n \times n)$ matrix can be upper triangular...

"Induction"

Eigenvalues: UT. every matrix.

$$U^{-1} M U = T$$

We proved last time:

M and T always have the same characteristic polynomial.

$$\det(M - \lambda I) = \det(T - \lambda I).$$

$\Rightarrow$ All e-values of M are e-vals of T

" " T " " M

Special property of T.

$$T = \begin{bmatrix} \lambda_1 & & \text{---} \\ & \lambda_2 \ddots & \text{---} \\ 0 & & \lambda_n \end{bmatrix}$$

For T, all entries along diagonal are the eigenvalues!.

To find e-valus of T, we want to find λ such that $T - \lambda I$ has a nullspace.

$$T = \begin{bmatrix} \lambda_1 & a_1 & a_2 \\ 0 & \lambda_2 & a_3 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$ 3x3 case.

$$\boxed{\underbrace{T-\lambda I}_{} = \begin{bmatrix} \lambda_1-\lambda & a_1 & a_2 \\ 0 & \lambda_2-\lambda & a_3 \\ 0 & 0 & \lambda_3-\lambda \end{bmatrix}}$$

$\lambda = \lambda_1$

$$T-\lambda_1 I = \underbrace{\begin{bmatrix} 0 & a_1 & a_2 \\ 0 & \lambda_2-\lambda_1 & a_3 \\ 0 & 0 & \lambda_3-\lambda_1 \end{bmatrix}}_{\text{HAS a NULLSPACE!}}$$

$T-\lambda_1 I$ has nullspace

$\Rightarrow \lambda_1$ must be an e-val of T!

Choose?

$\lambda = \lambda_2$

$$T - \lambda_2 I = \begin{bmatrix} \lambda_1 - \lambda_2 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & \lambda_3 - \lambda_2 \end{bmatrix}$$

$T - \lambda_2 I$ has nullspace

$\Rightarrow \lambda_2$ is an eig. values.

General case: $n \times n$ matrix.

$$\begin{bmatrix} \lambda_1 & & \text{stuff} & & \\ 0 & \lambda_2 & & & \\ & & \lambda_3 & & \\ & 0 & & \lambda_4 & \\ & & & & \lambda_5 \end{bmatrix} = T$$

Conside $T - \lambda_3 I$

$$\begin{bmatrix} \lambda_1 - \lambda_3 & & \text{stuff} & & \\ & \lambda_2 - \lambda_3 & & & \\ & & 0 & & \\ & 0 & & \lambda_4 - \lambda_3 & \\ & & & & \lambda_5 - \lambda_3 \end{bmatrix}$$

No pivot in col 3.

$\Rightarrow$ Free variable

$\Rightarrow$ Matrix is not invertible

$\Rightarrow$ It must have a nullspace.

BIBO stability if systems with non-diagonalizable matrices.

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}}_{\text{not diagonalizable}} \vec{x}[i] + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u[i]$$

Under what condition is this system BIBO stable?

$$\vec{x}[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}.$$

$$x_2[i+1] = \lambda \cdot x_2[i] + \beta \cdot u[i] \quad (1)$$

$\hookrightarrow$ is scalar sys? BIBO stable?

if $|\lambda| < 1$, then bounded $u \Rightarrow$ bounded x_2.

$$x_1[i+1] = \lambda x_1[i] + \underbrace{x_2[i] + \alpha u[i]}_{\text{general input}} \quad (2)$$

$$x_1[i+1] = \lambda x_1[i] + \text{input}$$

If $|\lambda| < 1$, is this BIBO stable?

Because $x_2[i]$ is bounded, we know input to (2) is bounded!

$\Rightarrow$ $x_1[i]$ is bounded!

$$\begin{bmatrix} \lambda_1 & & \text{stuff} & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \Rightarrow$$

If $|\lambda_i| < 1$ for all i then BIBO stable!

M : $\quad U^{-1} M U = T$

$$\underline{U}^T M \underline{U} = T \qquad U : \text{orthonormal!}$$

If M was symmetric.

$$M = M^T$$