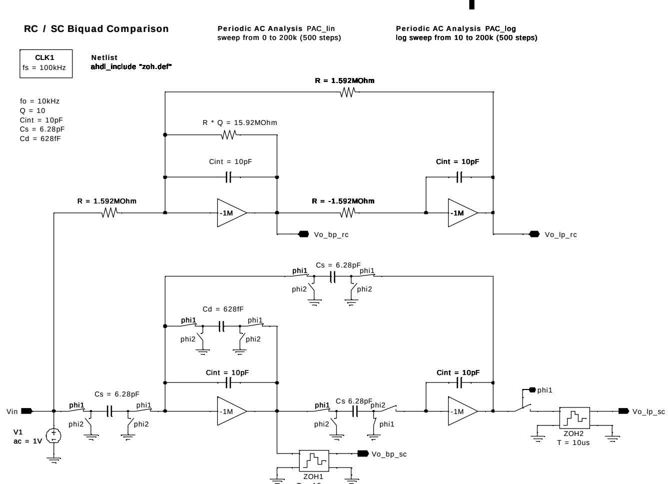
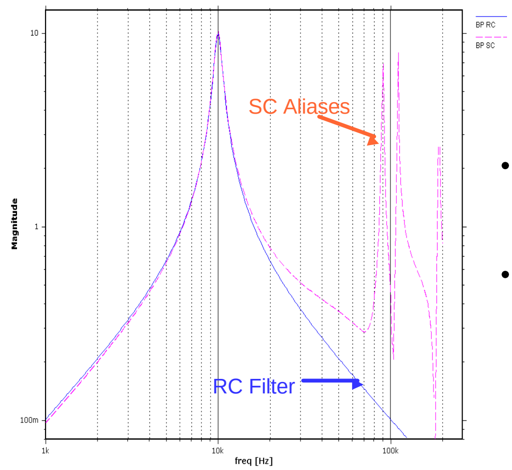
SC Filter Frequency Response

- Sampled data (and hence SC) filter responses are periodic
- CT and SD responses agree only for f << f_s
- ➢ Derive exact SD frequency response
  - RC – SC example
  - z-transform
  - SC integrator styles

RC / SC Bandpass

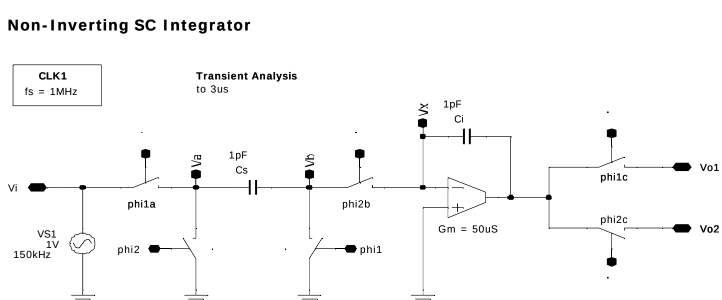


Frequency response

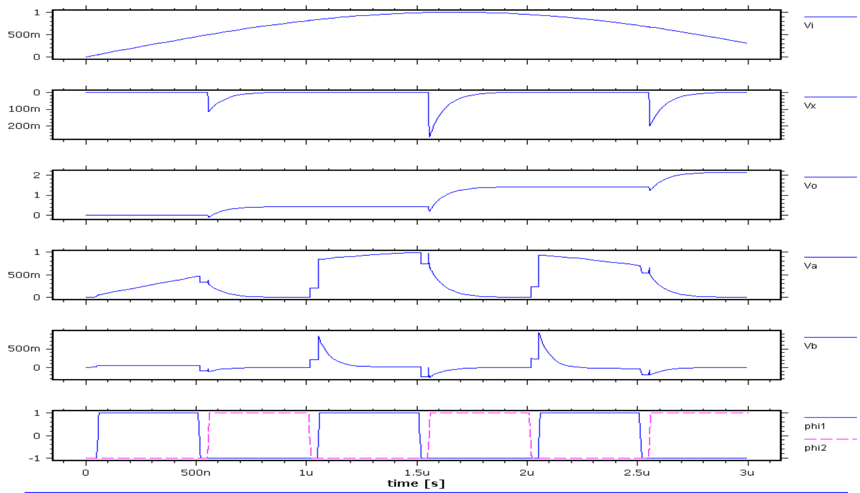


- CT bandpass has zero at f→infinity
- Where did that zero go in the SC filter?
  Note: $f_s/2$ would be a reasonable place!

Non-Inverting SC Integrator



Transient Analysis



Difference Equation

Φ_1 $Q_{s(kT)} = C_s V_{i(kT)}$ $Q_{i(kT)} = Q_{i(kT-T/2)}$

Φ_2 $Q_{s(kT+T/2)} = 0$ $Q_{i(kT+T/2)} = Q_{i(kT)} + Q_{s(kT)}$

$= Q_{i(kT-T/2)} + Q_{s(kT)}$

With $V_o = Q_i/C_i$ and $V_i = Q_s/C_s$ →

$$V_{o2(kT+T/2)} = V_{o2(kT-T/2)} + (C_s/C_i)\, V_{i(kT)}$$

Laplace Transform

$V_{o2(kT+T/2)} = V_{o2(kT-T/2)} + (C_s/C_i) * V_{i(kT)}$

Laplace Transform →

$$V_{o2}(s)e^{+s\frac{T}{2}} = V_{o2}(s)e^{-s\frac{T}{2}} + \frac{C_s}{C_i}V_i(s)$$

z-Transform

$$V_{o2}(s)e^{+s\frac{T}{2}} = V_{o2}(s)e^{-s\frac{T}{2}} + \frac{C_s}{C_i}V_i(s)$$

$$z \equiv e^{sT}$$

$$V_{o2}(z)z^{1/2} = V_{o2}(z)z^{-1/2} + \frac{C_s}{C_i}V_i(z)$$

$$V_{o2}(z) = V_{o2}(z)z^{-1} + \frac{C_s}{C_i}V_i(z)z^{-1/2}$$

$$H_{\text{int\_ non-inverting\_2}}(z) = \frac{V_{o2}(z)}{V_i(z)} = \frac{C_s}{C_i}\frac{z^{-1/2}}{1-z^{-1}}$$

$$H_{\text{int\_ non-inverting\_1}}(z) = \frac{V_{o1}(z)}{V_i(z)} = \frac{C_s}{C_i}\frac{z^{-1}}{1-z^{-1}}$$

Ref: Oppenheim and Schafer, *Discrete-Time Signal Processing*, Chapter 4

Note: the derivation assumes that the output is taken at the end of phase 2. If, as is often the case, the output is used only at the end of the next phase 1, the numerator is z^{-1}.

Frequency Response

$$H_{\mathrm{SC\_ni}}(f) = H_{\mathrm{int\_ni}}(z)\Big|_{z=e^{2pjfT}}$$

$$= \frac{C_s}{C_{\mathrm{int}}}\frac{z^{-1/2}}{1-z^{-1}}\Bigg|_{z=e^{2pjfT}} = \frac{C_s}{C_{\mathrm{int}}}\frac{1}{z^{1/2}-z^{-1/2}}\Bigg|_{z^{1/2}=e^{pjfT}=\cos pfT + j\sin pfT}$$

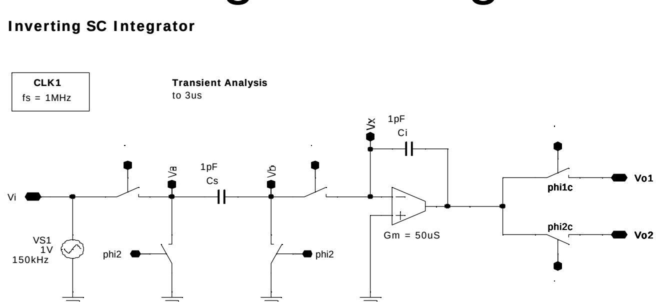
$$= \frac{C_s}{C_{\mathrm{int}}}\frac{1}{\cos pfT + j\sin pfT - \cos pfT + j\sin pfT}$$

$$= \frac{C_s}{C_{\mathrm{int}}}\frac{1}{2j\sin pfT}$$

$$\approx \frac{C_s}{C_{\mathrm{int}}}\frac{1}{2pjfT} \qquad \text{for} \qquad fT \ll 1 \quad \text{or} \quad f \ll f_s$$

$$H_{\mathrm{RC}}(f) = \frac{1}{2pjfRC}$$

Inverting SC Integrator

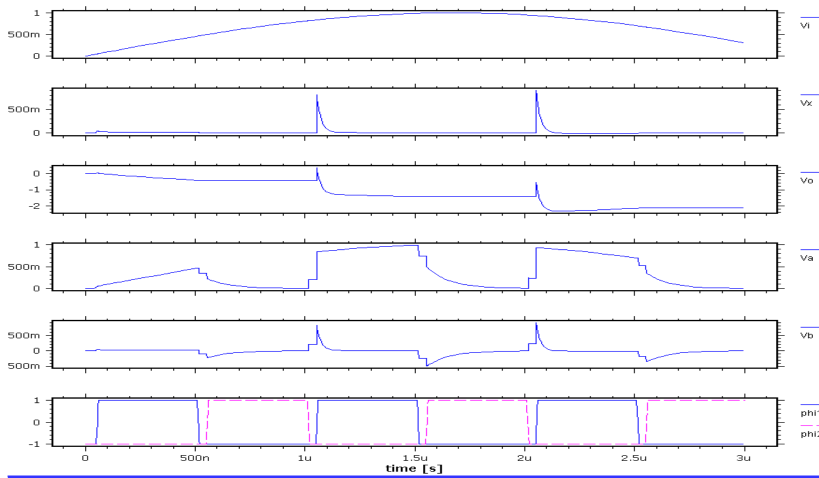


$$H_{\mathrm{int\_i}}(z) = -\frac{C_s}{C_{\mathrm{int}}}\frac{z^{-1/2}}{1-z^{-1}}$$

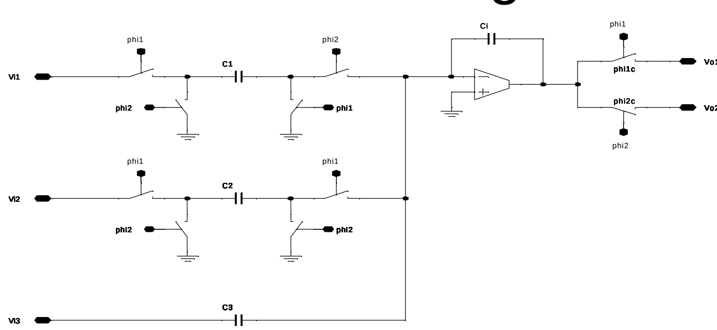
$$H_{\mathrm{SC\_i}}(f) = -\frac{C_s}{C_{\mathrm{int}}}\frac{1}{2j\sin pfT}$$

Note: $H_{int\_i}(z)$ assumes that the output is used at the end of phase 2. If, as is often the case, the output is used already at the end of phase 1, the numerator is 1.
If, as is normally the case, circuit topologies are used where inverting and non-inverting SC integrators alternate, this is not an issue.

Transient Analysis



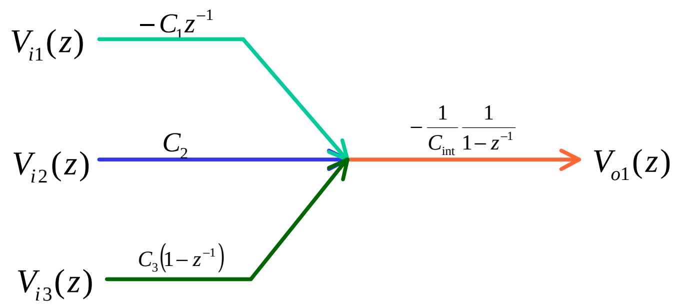
General SC Integrator



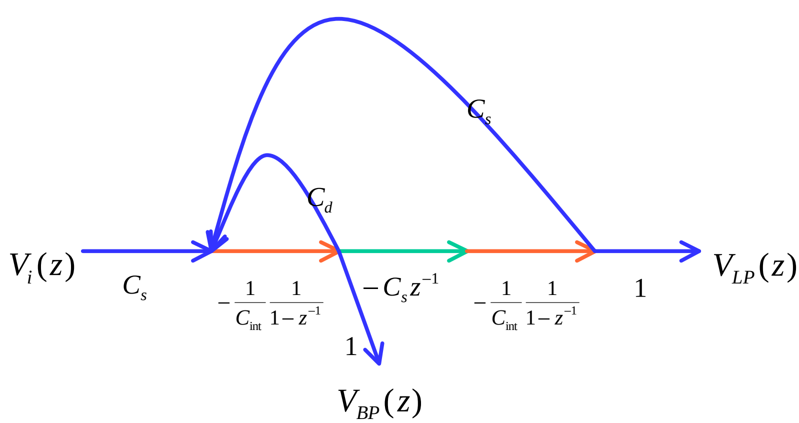
$$V_{o1}(z)=\frac{C_1}{C_i}\frac{z^{-1}}{1-z^{-1}}V_{i1}(z)-\frac{C_2}{C_i}\frac{1}{1-z^{-1}}V_{i2}(z)-\frac{C_3}{C_i}V_{i3}(z)$$

$$V_{o2}(z)=\frac{C_1}{C_i}\frac{z^{-1/2}}{1-z^{-1}}V_{i1}(z)-\frac{C_2}{C_i}\frac{z^{-1/2}}{1-z^{-1}}V_{i2}(z)-\frac{C_3}{C_i}V_{i3}(z)$$

Integrator Signal Flow Diagram



SFD of SC BP



Frequency Response of SC BP

$$V_{BP} = -\frac{1}{C_{\text{int}}}\frac{1}{1-z^{-1}}\left(C_s V_i + C_s V_{LP} + C_d V_{BP}\right)$$

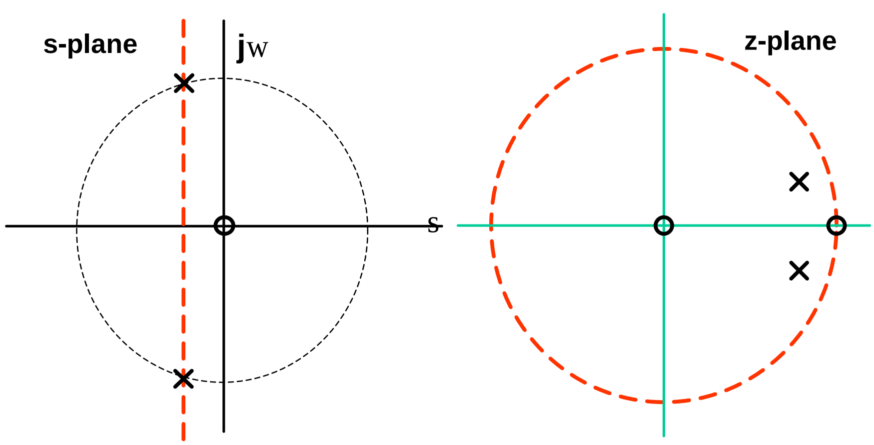
$$V_{LP} = -\frac{1}{C_{\text{int}}}\frac{z^{-1}}{1-z^{-1}}\left(-C_s V_{BP}\right)$$

some work (Mason's rule helps)

$$\text{H}_{\text{BP}} = \frac{V_{BP}}{V_i} = \frac{z(z-1)}{z^2\left(\frac{f_s}{w_0}+\frac{w_0}{f_s}+\frac{1}{Q}\right) - z\left(2\frac{f_s}{w_0}+\frac{1}{Q}\right) + \frac{f_s}{w_0}}$$

$$\text{using: } \frac{C_s}{C_{\text{int}}} = \frac{w_0}{f_s} \quad \text{and} \quad \frac{C_s}{C_d} = \frac{1}{Q}$$

s-Plane versus z-Plane

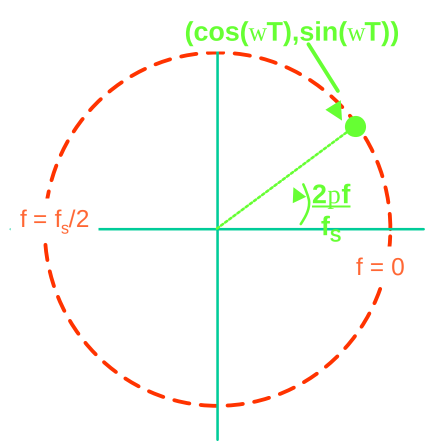


z-Plane

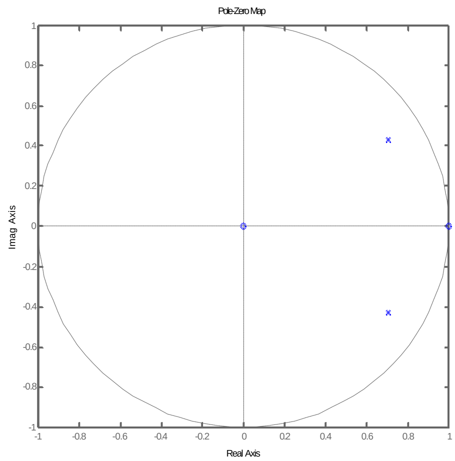
- In the z-plane
  - The distance from the pole to the unit circle is inversely proportional to pole Q
  - The angle to the pole is equal to 360° (or 2π radians) times the ratio of the pole frequency to the sampling frequency
- How do poles and zeroes in the complex z-plane relate to frequency response?

z-Domain Frequency Response

- The jω axis maps onto the unit circle
- Particular values:
  - $f = 0 \rightarrow z = 1$
  - $f = f_s/2 \rightarrow z = -1$
- The frequency response is obtained by evaluating H(z) on the unit circle at $z = e^{j\omega T} = \cos(\omega T)+j\sin(\omega T)$
- Once z=1 ($f_s/2$) is reached, the frequency response repeats, as expected

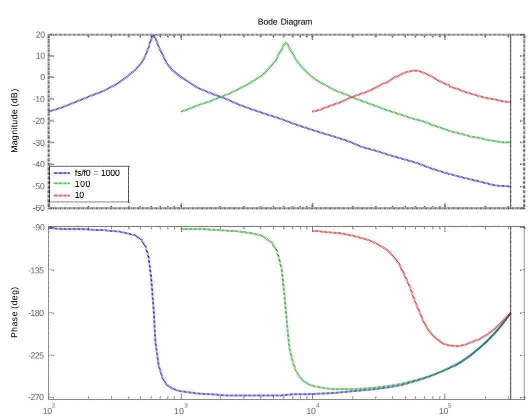


Pole-Zero Map in z-Plane



The zero from f→infinity is mapped to z=0, a non-physical frequency. This explains the poor attenuation at $f=f_s/2$.

Frequency Response



Frequency Warping

- Frequency response
  - Continuous time (s-plane): imaginary axis
  - Sampled time (z-plane): unit circle
- Continuous to sampled time transformation
  - Should map imaginary axis onto unit circle
  - How do SC integrators map frequencies?

$$H_{SC}(z) = \frac{C_s}{C_{int}} \frac{z^{-1/2}}{1 - z^{-1}}$$
$$= -\frac{C_s}{C_{int}} \frac{1}{2j \sin p f T}$$

CT – SC Integrator Comparison

CT Integrator

$$H_{RC}(s) = -\frac{1}{st}$$
$$= -\frac{1}{2pjf_{RC}t}$$

SC Integrator

$$H_{SC}(z) = \frac{C_s}{C_{int}} \frac{z^{-1/2}}{1 - z^{-1}}$$
$$= -\frac{C_s}{C_{int}} \frac{1}{2j \sin p f_{SC} T}$$

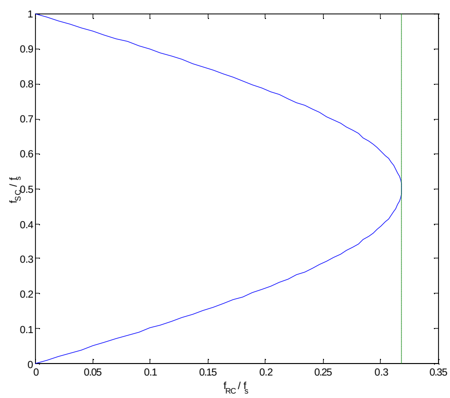
Identical time constants:

$$t = RC = \frac{C_{int}}{f_s C_s}$$

Compare: $H_{RC}(f_{RC}) = H_{SC}(f_{SC})$ →

$$f_{RC} = \frac{f_s}{p} \sin\left(p \frac{f_{SC}}{f_s}\right)$$

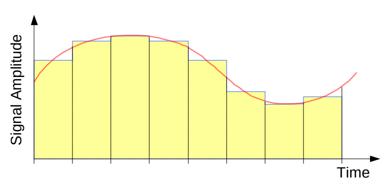
LDI Integration



$$f_{RC} = \frac{f_s}{p}\sin\left(p\,\frac{f_{SC}}{f_s}\right)$$

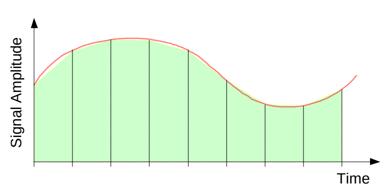
- "RC" frequencies up to f_s/π map to physical (real) "SC" frequencies
- Frequencies above f_s/π do not map to physical frequencies
- Mapping is symmetric about $f_s/2$ (aliasing)
- "Accurate" only for $f_{RC} << f_s$

Types of Integration



Lossless discrete integration

$$H(z) = \frac{z^{-1/2}}{1 - z^{-1}}$$



Bilinear integration ("Delta rule")

$$v_o(kT) = v_o(kT - T) + \frac{T}{2}\left[v_i(kT) + v_i(kT - T)\right]$$

(and many others, e.g. Euler, Runge Kutta, Gear, …)

Bilinear Transform

- Bilinear integrator

$$v_o(kT) = v_o(kT-T) + \frac{T}{2}\left[v_i(kT) + v_i(kT-T)\right]$$

$$\left[1 - z^{-1}\right]V_o(z) = \frac{T}{2}\left[1 + z^{-1}\right]V_i(z)$$

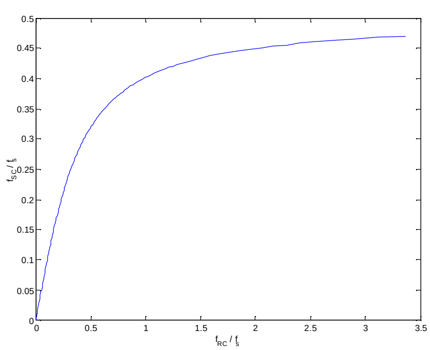
$$H(z) = \frac{V_o(z)}{V_i(z)} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

- Frequency translation

$$\left.\frac{1}{s}\right|_{s=2pjf_{RC}} = \left.H_{SC}(z)\right|_{z=e^{2pjf_{SC}T}}$$

$$f_{RC} = \frac{f_s}{p}\tan\left(p\,\frac{f_{SC}}{f_s}\right)$$

Bilinear Transform



$$f_{RC} = \frac{f_s}{p}\tan\left(p\,\frac{f_{SC}}{f_s}\right)$$

- Entire jω axis maps onto the unit circle
- Mapping is nonlinear (tan distortion)
  → prewarp specifications of RC prototype
  Matlab filter design automates this (see, e.g. `bilinear`)

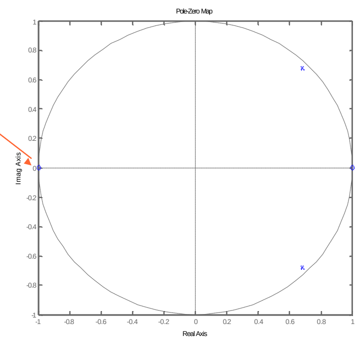
"Bilinear" Bandpass

$f_s = 100\text{kHz}$

$f_c = f_s/8$

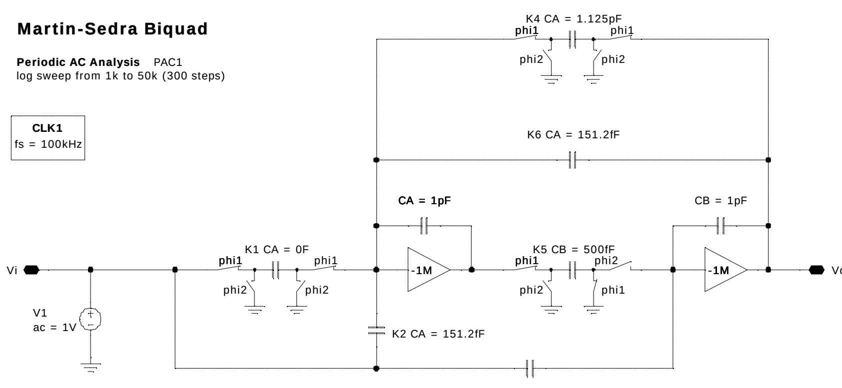
$Q = 10$

zero at $f_s/2$



Matlab:

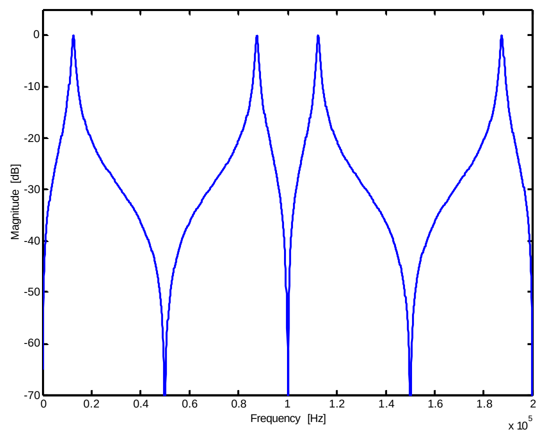
```
           0.0378 z^2 - 0.0378
H(z) = ---------------------
         z^2 - 1.362 z + 0.9244
```

$$\frac{V_o(z)}{V_i(z)} = \frac{K_3 z^2 + (-2K_3 + K_1K_5 + K_2K_5)z + (K_3 - K_2K_5)}{z^2 + (-2 + K_4K_5 + K_5K_6)z + (1 - K_5K_6)}$$

Ref: K. Martin and A. S. Sedra, "Strays-insensitive switched-capacitor filters based on the bilinear z transform," Electron. Lett., vol. 19, pp. 365-6, June 1979.

Magnitude Response



LD vs Bilinear Transform

- LDI transform:
  - Realized by “standard” SC integrators
  - High frequency zeros are lost
  - Simple filter synthesis:
    - Replace RC integrators with SC integrators
    - Ensure that inverting and non-inverting integrators alternate in loops
- Bilinear transform
  - Maps entire jω axis onto unit circle (nonlinear mapping)
  - Not implemented by “standard” SC integrators
  - Synthesis:
    - Biquads: direct coefficient comparison
    - Ladders: see
      R. B. Datar and A. S. Sedra, “Exact design of strays-insensitive switched capacitor high-pass ladder filters”, Electron. Lett., vol. 19, no 29, pp. 1010-12, Nov. 1983.